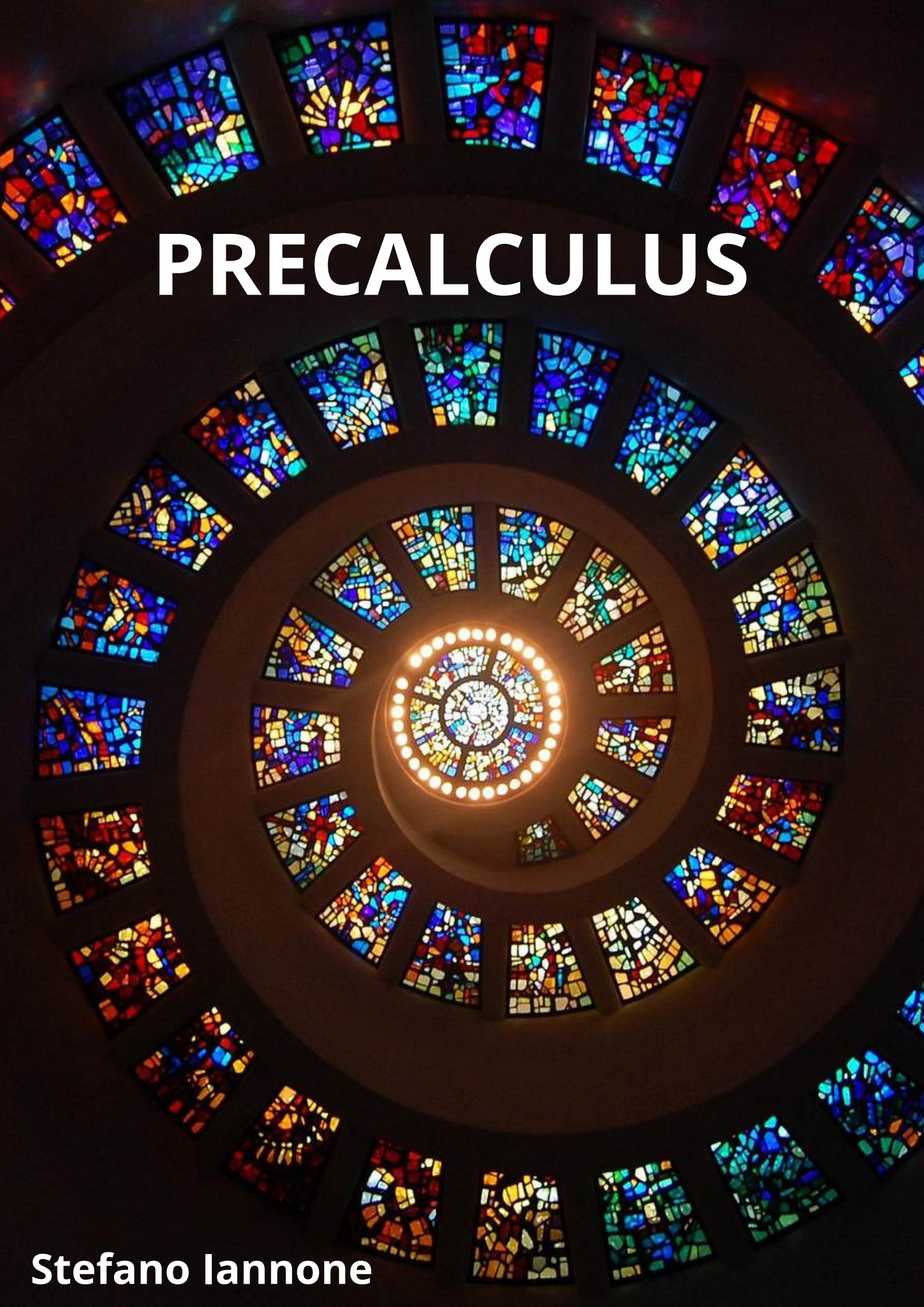
PRECALCULUS



Stefano Iannone

Note for readers and colleagues:
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ABOUT THE AUTHOR

Stefano Iannone, born in Rome in 1959, cultivated two great passions from a young age: mathematics and basketball. After 16 years as a professional basketball player (Series A, B, and C), he brought the same discipline and determination to the study of mathematics. He trained in both Italy and the United States, earning a degree in Physical Education in Italy and later a degree in Applied Mathematics at San Francisco State University. Upon returning to Rome, he obtained a Master's degree in "Codes and Numbers" from the University of Rome Tor Vergata. From 1998 to 2022, he taught mathematics at Marymount International School in Rome, and since 2009 he has been an adjunct professor at John Cabot University in Rome. Beyond teaching, he enjoys writing about mathematics, striving to show how this discipline is present in our daily lives. He firmly believes that mathematics, much like sports, can be a powerful tool for social inclusion and personal growth. Married since 1987, he is the father of three children and grandfather to a wonderful 5-year-old grandson and a wonderful newborn granddaughter.

INTRODUCTION

As a university mathematics instructor, I have spent many years helping students make the transition from high school mathematics to university-level courses. Throughout that time, I have recognized the need for a comprehensive textbook for a one-semester Precalculus course. This manual was written to provide students with a clear, complete, and accessible resource covering the essential course topics. University Precalculus classrooms are wonderfully diverse. Students arrive with different educational backgrounds, varying levels of mathematical preparation, and different degrees of confidence. This manual grew out of the notes I developed over many years of teaching students with these diverse needs. It is designed to help those who need to strengthen their mathematical foundations while also providing the depth necessary to prepare every student for the challenges of calculus. Above all, I believe that mathematics should not feel like a series of arbitrary obstacles. My goal is to help students appreciate mathematics not only for its practical applications but also for its elegance, logical structure, and beauty.

Methodology and Organization

This book covers the standard university Precalculus curriculum. The topics are presented in a carefully planned sequence so that each chapter builds naturally on the previous one, allowing students to develop increasingly sophisticated mathematical ideas with confidence.

The presentation is rigorous enough to ensure mathematical accuracy without becoming unnecessarily formal or burdened by lengthy theoretical proofs. Since this course is intended primarily for students outside mathematics, the emphasis is placed on understanding fundamental concepts and developing the problem-solving techniques needed for further study and real-world applications.

Features of This Manual

One of the strengths of this text is its extensive collection of exercises, developed and refined through many years of teaching courses ranging from algebra to calculus.

- **Worked Examples:** Every chapter includes carefully chosen, step-by-step examples that illustrate important concepts and problem-solving strategies.
- **Extensive Practice:** Each section contains a wide variety of exercises, progressing from routine applications to more challenging problems that encourage deeper understanding.
- **Exam Preparation:** Many exercises are modeled after the types of questions commonly found on university quizzes, midterm examinations, and final exams. They are designed to strengthen both computational skills and conceptual reasoning.

Mathematics is a collaborative endeavor, and no textbook is ever truly finished. I warmly welcome suggestions, comments, and constructive criticism from both students and colleagues. Your feedback will help improve future editions of this manual.

Finally, I would like to express my deepest gratitude to my wife for her endless patience and encouragement throughout the writing of this book. I am also sincerely grateful to my colleagues for their support and to the many students whose questions, curiosity, and perseverance have inspired and improved this work. Thank you all.

Stefano Iannone
Summer 2026

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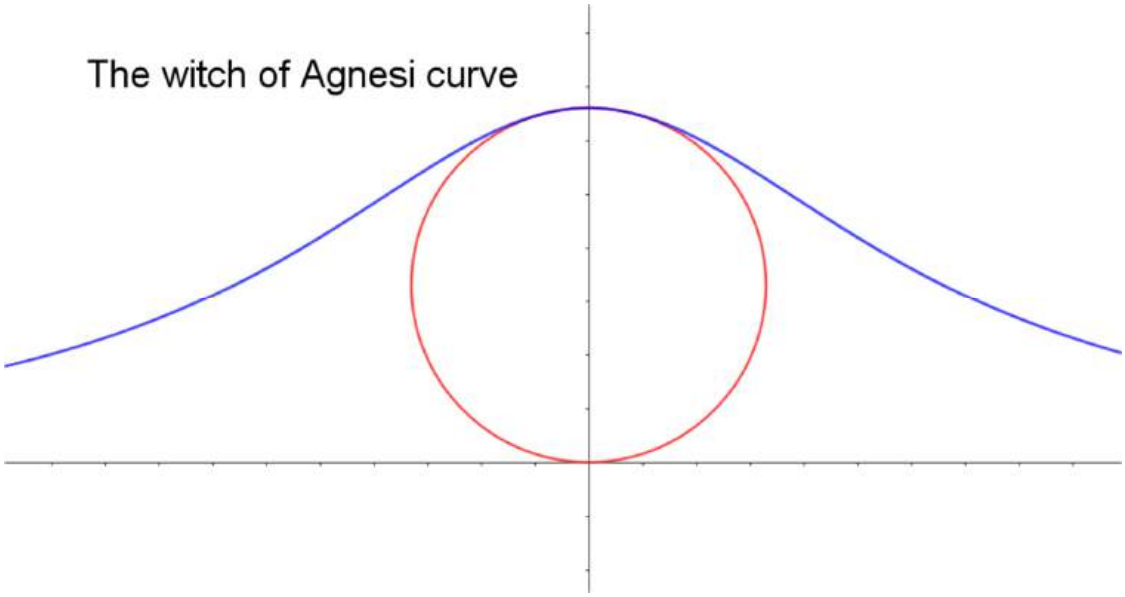
CHAPTER 1 – Functions

"Analysis is the infinitely difficult art of extracting information from an unknown, given only by an equation." **Leonhard Euler** *(The master who popularized the notation* $f(x)$*)*



1.1 The Fundamental Concept of a Function

Topics: Domain, Range, Interval Notation

In precalculus, it's easy to mistake a function for just an equation you plug numbers into. But in higher mathematics, a function is much deeper: it is a precise bridge that connects two sets of numbers, *an input and an output*. By focusing on this relationship, what goes in and what comes out, and *why,* you build a mental map of how quantities change together. Mastering this view will prevent common structural errors later, where the "formula" isn't always the whole story.

Formal Definition of a Function

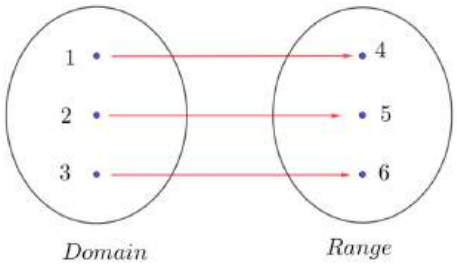
A function f from a set D to a set R is a relation that assigns to each element x in the set A exactly one element y in the set B. The set D is called the domain, and the set R contains the range.

Notation and Evaluation of a Function

Function notation uses symbols like $f(x) =$ or $y =$, which means "the value of the function f at input x." To evaluate a function, you replace x with a given number and simplify to find the result.

The Input-Output Mapping View

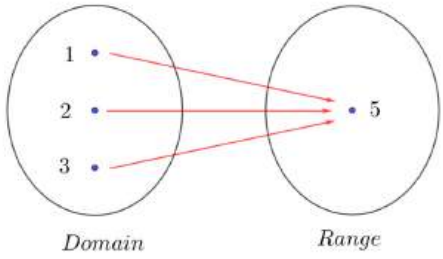
To visualize this, imagine the domain as a set of unique keys and the range as a set of locked boxes. A valid function means that every single key opens exactly one specific box. If a single key could suddenly open two completely different boxes, the relation fails to be a function—it is mathematically ambiguous.



$function\ \ one-to-one$

Function one to one:

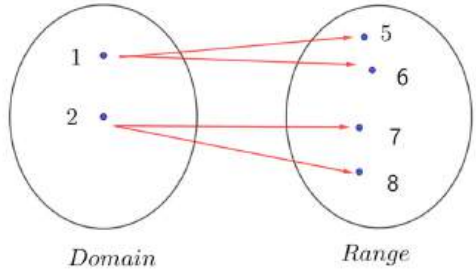
at each element of the domain corresponds exactly one element of the other set and viceversa



$function\ \ many-to-one$

Function many to one:

each element of the domain corresponds one and only one element of the other set.



$not-a-function$

Function many to one:

each element of the domain does not correspond to exactly one element of the other set.

Ordered Pairs

Another way to determine if a relation is a function is using the concept of *ordered pairs*.

A function can also be defined as a relation in which different ordered pairs have different first coordinates.

Evaluating a Function

To evaluate a function, simply substitute the given input value into the function's equation wherever that variable appears, then simplify using the standard order of operations.

Example 1:

$f(x)=x+2$ it is a function because for every input value of x, there is exactly one output.

If $x=3$ then $f(x)=3+2=5$

Example 2:

$f(x)=x^2$ it is a function because each input value of x produces only one output, even though different inputs can give the same result.

If $x=3$ then $f(3)=(3)^2=9$

If $x=-3$ then $f(-3)=(-3)^2=9$

This is still fine because each input has only one output.

Example 3:

$y^2=x^2+1$ it is **NOT** a function because one input value of $x$can correspond to **more than one output**.

If $x=0$ then $y^2=1\rightarrow y=-1$ or $y=1$

Example 4:

The set of ordered pairs $\{(1,2)(2,3)(3,6)\}$ is a function even if no rule is given.

Example 5:

The set of ordered pairs $\{(1,3)(1,4)(2,5)\}$ is a relation but NOT a function, since $(1,3)$ and $(1,4)$ are different pairs with the same first coordinate.

Example 6:

Evaluate $f(x)=x^3-x+3,$ when $x=-1$.

Solution:

Replace x with -1: $f(-1)=(-1)^3-(-1)+3=-1+1+3=3$

Example 7:

Given $f(x)=x^2,$ find $f(a+1)$

Solution:

Step 1: Replace x with $a+1$.

Step 2: Then we have $f(a+1)=(a+1)^2$

Step 3: Expand $f(a+1)=(a+1)^2=a^2+2a+1$

Example 8:

Given $f(x)=x^2-x,$ find $f(x+h)$

Solution:

Step 1: Replace x with $x+h$.

Step 2: Then we have $f(x+h)=(x+h)^2-(x+h)$

Step 3: Expand $f(x+h)=(x+h)^2-(x+h)=x^2+2xh+h^2-x-h$

Domain and Range Defined

Definitions of Domain and Range

The Domain is the complete set of all possible valid input values (typically x-values) for which the function is defined and produces a real number as an output.

The Range is the complete set of all resulting output values (typically y-values or f(x)) that the function can physically output after running every number in the domain through its mechanism.

Finding the domain of a function, there are two major constraints: *division by zero and taking even roots of negative numbers.*

Excluding Division by Zero for a Rational Function: Because division by zero is undefined, any input that causes a denominator to vanish must be strictly excluded from the domain:

Given $f(x)=\frac{N(x)}{D(x)}$ where $N(x)$ is the numerator and $D(x)$ is the denominator, you must enforce the condition that $D(x)\neq 0$ and solve for x to find the restrictions.

Avoiding Even Roots of Negative Numbers: In the real number system, you cannot take the square root, fourth root, or any even root of a negative value. Therefore, for a radical function like $f(x)=\sqrt[2n]{g(x)}$, you must ensure the radicand is non-negative by setting up and solving the inequality $g(x)\geq 0$.

Example 9:

Let $f(x)=\frac{x}{2x-4}$,find the domain.

Solution:

$2x-4$ cannot equal 0. Therefore $2x-4\neq 0\rightarrow x\neq 2$.

Example 10:

Let $f(x)=\sqrt{x+3}$, find the domain.

Solution:

$x+3$ must be greater or equal to 0. Therefore $x+3\geq 0 \rightarrow x\geq -3$.

Interval Notation Essentials

In mathematics, to describe a set of boundaries we use a streamlined shorthand called interval notation using parentheses and brackets.

Inequality Notation	Interval Notation	Geometric Graphic Meaning
$a<x<b$	(a,b)	Open interval: Excludes both endpoints *a* and *b*.
$a\leq x\leq b$	$[a,b]$	Closed interval: Includes both endpoints *a* and *b*.
$a\leq x<b$	$[a,b)$	Half-open interval: Includes *a*, excludes *b*.
$x\geq a$	$[a,\infty)$	Infinite interval: Starts at *a* and goes to infinity. (Infinity always takes a parenthesis).

Example 11: Rational Function Domain

Find the domain of the function: $f(x)=\dfrac{2x+1}{x^2-4x-12}$

Solution:

Step 1: Identify the restriction. This is a rational function, so the denominator cannot be zero.

Step 2: Set the denominator equal to zero and solve for the excluded values:

$x^2-4x-12=0\rightarrow (x-6)(x+2)=0$

Therefore, $x=6$ or $x(-2)$.

Step 3: Exclude these values from the real number line. The domain consists of all real numbers except -2 and 6.

Step 4: Formulate into interval notation using the union symbol $(\cup)$:

Domain = $(-\infty,-2)\cup(-2,6)\cup(6,\infty)$

Example 12: Radical Function Domain

Find the domain of the function: $g(x)=\sqrt{5-2x}$

Solution:

Step 1: Identify the restriction. This is an even root, so the radicand must be greater than or equal to zero.

Step 2: Set up the inequality: $5-2x\geq 0$

Step 3: Isolate *x*. Remember that multiplying or dividing an inequality by a negative number flips the inequality sign: $2x\geq -5\rightarrow x\leq \dfrac{5}{2}$

Step 4: Convert this boundary condition into interval notation:

Domain = $\left(-\infty,\dfrac{5}{2}\right]$

Example 13: Combined Restrictions

Find the analytical domain of the function: $f(x)=\dfrac{\sqrt{x}}{(x-3)}$

Solution:

Step 1: Evaluate the radical in the numerator. The expression inside the square root requires $x \geq 0$.

Step 2: Evaluate the denominator. The expression cannot be zero, meaning $x-3 \neq 0 \rightarrow x \neq 3$.

Step 3: Combine both constraints. We need values of x that are greater than or equal to 0, but absolutely cannot equal 3.

Step 4: Map this compound condition onto the number line and convert to interval notation:
Domain = $[0,3) \cup (3,\infty)$

Finding the range is often trickier than finding the domain because you are looking for all the possible *output* values (*y-values*) rather than the inputs. Generally there are two primary strategies:

Finding the Range:

The Inverse Method (Algebraic Isolation): For many functions, the most direct route is to rewrite the function as $y=f(x)$ and algebraically solve for x in terms of y. (*This effectively turns the problem inside out, allowing you to find the range of the original function by finding the domain of its inverse.*) You simply look for the same constraints: *division by zero and even roots*. But this time, you restrict the *y*-values.

Analyzing Geometric and Algebraic Bounds: Some functions have inherent, structural limitations. For instance, because a squared term or an even radical can never yield a negative result in the real number system, you know that $x^2 \geq 0$ and $\sqrt{x} \geq 0$. You can use these fundamental inequalities as a starting point, building the rest of the function around them to establish the upper or lower bounds of the range.

Example 14: Rational Function (Range)

Find the range of $f(x)=\dfrac{3x-5}{2x+1}$

Solution:

Step 1: write as $y=\dfrac{3x-5}{2x+1}$

Step 2: change x and $y \rightarrow x=\dfrac{3y-5}{2y+1}$

Step 3:solve for *y*:

$\rightarrow x(2y+1)=3y-5 \rightarrow 2xy+x=3y-5 \rightarrow 2xy-3y=-5-x \rightarrow y(2x-3)=-5-x \rightarrow y=\dfrac{-5-x}{2x-3}$

Step 4: The domain of this function is $2x-3\neq 0 \rightarrow x\neq \dfrac{3}{2}$

Step 5: therefore the range is: $y\neq \dfrac{3}{2}$ or Range: $\left(-\infty,\dfrac{3}{2}\right)\cup\left(\dfrac{3}{2},\infty\right)$

Example 15: Root Function (Range)

Find the range of $f(x)=4-\sqrt{2x+6}$

Solution:

Because $f(x)\leq 4$ and the radical can grow infinitely large as *x* approaches infinity $(x\rightarrow\infty)$, the output values start at 4 and drop down to negative infinity. Therefore Range: $(-\infty,4]$.

Miscellaneous Problems

Example 16:

Find the value(s) of *a* such that the domain of $f(x)=\dfrac{\sqrt{x-a}}{\sqrt{x-4}}$ is $(4,\infty)$

Solution:

$x>4$ from denominator, $x\geq a$ from numerator. For domain = $(4,\infty)$ we need $a=4$.

Example 17:

Find domain of $f(x)=\sqrt{x^2-4}+\dfrac{1}{\sqrt{9-x^2}}$

Solution:

First term: $x^2-4\geq 0\rightarrow x\leq -2$ or $x\geq 2$

Second term: $9-x^2>0\rightarrow -3<x<3$

Intersection: $(-3,-2]\cup[2,3)$

Example 18:

If the domain of $f(x)=\dfrac{\sqrt{x+a}}{x^2-5x+6}$ is $(-3,-2]\cup(2,3)\cup(3,\infty)$, find a.

Solution:

Denominator: $x^2-5x+6\neq 0\rightarrow x\neq 2, x\neq 3$

numerator: $x+a\geq 0\rightarrow x\geq -a$

Given domain starts at $-3\rightarrow -a=-3\rightarrow a=3$

Example 19:

If the domain of $f(x)=\dfrac{x+a}{x+b}$ is $x\neq 2$, find a and b if $f(x)$ passes through (0, - 2).

Solution:

Domain: $x\neq 2\rightarrow x+b\neq 0\rightarrow b=-2$

$f(0)=-2\rightarrow \dfrac{a}{0-2}\rightarrow a=4$

Practice Problems

For each function find the indicated values

1. $f(x)=x^2+2;\ f(3)$ $[11]$
2. $f(x)=\frac{1}{2}x^2+1;\ f(1)$ $\left[\frac{3}{2}\right]$
3. $f(t)=t+2;\ f\left(-\frac{1}{4}\right)$ $\left[\frac{7}{4}\right]$
4. $f(t)=|t+3|;\ f(-10)$ $[13]$
5. $h(x)=x^3+4x^2;\ h(-2)$ $[8]$
6. $g(a)=a^2+\frac{1}{5}a;\ g(-1)$ $\left[\frac{4}{5}\right]$
7. $g(x)=-2x^3-\frac{1}{5}x^2;\ g\left(-\frac{1}{3}\right)$ $\left[\frac{7}{135}\right]$
8. $f(x)=x+2;\ f(a+1)$ $[a+3]$
9. $f(x)=x^2-1;\ f(a+3)$ $[a^2+6a+8]$
10. $f(x)=\frac{1}{x};\ f(x+h)$ $\left[-\frac{h}{x^2+xh}\right]$
11. $f(x)=2x^2-3;\ \frac{f(x+h)-f(x)}{h}$ $[4x+2h]$
12. $f(x)=x^2+6;\ \frac{f(x+h)-f(x)}{h}$ $[2x+h]$
13. $f(x)=\sqrt{x};\ \frac{f(x+h)-f(x)}{h}$ $\left[\frac{1}{\sqrt{x+h}+\sqrt{x}}\right]$
14. Given $f(x)=x^2+1$, find
    (i) $f(x+2)$ $[x^2+4x+5]$
    (ii) $f(x)+f(2)$ $[x^2+6]$
15. Given $f(x)=3x-1$, find
    (i) $f(2x)$ $[6x-1]$
    (ii) $2f(x)$ $[6x-2]$
16. Given $f(x+3)=\frac{3f(x)+9}{6}$ such that $f(5)=4$, find the value of $f(2)$ $[5]$
17. Given $f(x+1)=\frac{2f(x)+1}{2}$ such that $f(2)=2$, find the value of $f(1)$ $\left[\frac{3}{2}\right]$

18. Given $f(x-1)=3f(x)-2$ such that $f(1)=5$, find the value of $f(3)$ $\left[\frac{13}{9}\right]$

19. Given $f(x+1)=\frac{xf(x)}{3}$ such that $f(3)=18$, find the value of $f(1)$ $[81]$

Find the domain of the function

20. $f(x)=\frac{x-1}{x^2+3x}$ $[x\neq 0,-3]$

21. $f(x)=\sqrt{x^2-4}$ $[|x|\geq 2]$

22. $f(x)=\frac{\sqrt{4x^2-1}}{x^2-9}$ $\left[|x|\geq\frac{1}{2}, x\neq\pm 3\right]$

23. $f(x)=\sqrt{\frac{2-x}{x+6}}$ $[-6<x\leq 2]$

24. $f(x)=\frac{\sqrt{2x-1}}{3x^2-1}$ $\left[\frac{1}{2}\leq x\leq\frac{\sqrt{3}}{3}, x>\frac{\sqrt{3}}{3}\right]$

25. Given $y=\frac{2}{ax^2+2x+5}$, find the value of a so the domain is $x\neq -5$ $\left[a=\frac{1}{5}\right]$

Find the range of the function

26. $y=\frac{2-x}{x}$ $[y\neq -1]$

27. $f(x)=\sqrt{9-x^2}$ $[0\leq y\leq 3]$

28. $f(x)=\frac{4}{x^2+2}$ $[0<y\leq 2]$

29. $f(x)=5-\sqrt{3x-1}$ $[-\infty<y\leq 5]$

30. $f(x)=\frac{x}{x^2+4}$ $\left[-\frac{1}{4}\leq y\leq\frac{1}{4}\right]$

1.2 -Analyzing Graph Behavior

Topic: Graph Behavior, Intercepts, Symmetry, and Transformations

Review

Evaluate each function.

1) $h(n) = |n-3|$; Find $h(4n)$
$|4n-3|$

2) $g(x) = x^3 - 4x^2$; Find $g(4x)$
$64x^3 - 64x^2$

3) $p(a) = 2^{a+3}$; Find $p(3a)$
2^{3a+3}

4) $h(n) = -3 + \frac{5}{4}n$; Find $h\left(\frac{3n}{4}\right)$ $-3 + \frac{15}{16}n$

5) Given $f(x) = x^2 - 3$ evaluate
(i) $f(x+4)$ (i) $x^2 + 8x + 13$
(ii) $f(x) + f(4)$ (ii) $x^2 + 10$

6) Evaluate $\frac{f(x+h) - f(x)}{h}$ for
(i) $f(x) = 3x + 2$
(ii) $f(x) = x^2 + 1$

7) Given the function $f(x) = x^2 - 3$ find $\frac{f(x+h) - f(x)}{h}$
$2x + h$

8) Given a function $f(x+1) = \frac{3f(x) - 2}{2}$ such that $f(3) = 1$, find the value of $f(2)$.
$f(2) = 2$

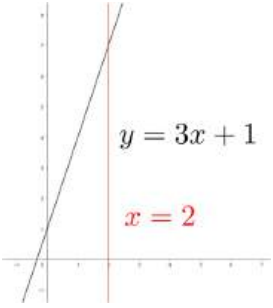
The Vertical Line Test

Another method for visualizing a function is its *graph*. The graph of *f* consists of all points *(x, y)* in the coordinate plane such that $y = f(x)$ and *x* is in the domain of *f*. Thus the graph of a function f is the same as the graph of the equation $y = f(x)$. One question arises: which curves in the *xy* plane are graphs of functions?

> *A curve in the plane is the graph of a function if and only if no vertical line intersects the curve more than once.*

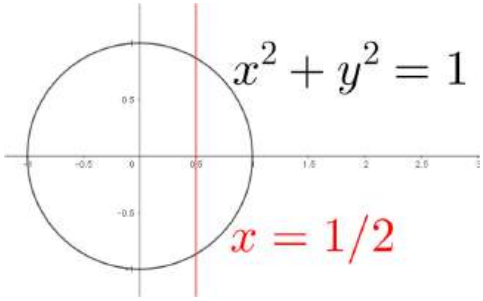
Example 1:

Line $y = 3x + 1$ is a graph of a function because a vertical line intersects the curve only once.



Example 2:

The equation of a circle $x^2 + y^2 = 1$ is **NOT** a graph of a function because a vertical line intersects the curve more than once.



Intercepts

> • *x-intercepts: The points where a graph intersects the x-axis. They occur where y = 0 or f(x) = 0. In algebra, these are also explicitly known as the roots, zeros, or solutions of the function.*
>
> • *y-intercept: The point where a graph crosses the y-axis. It occurs where x = 0. A valid function can have at most one y-intercept, according to the Vertical Line Test.*

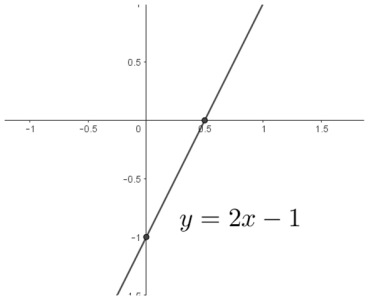
Example 3:

Find the x and y intercepts of $y = 2x - 1$, then graph the line

Solution:

x-intercept when $y = 0$: $2x - 1 = 0 \rightarrow x = \frac{1}{2}$

y-intercept when $x = 0$: $y = 2(0) - 1 \rightarrow y = -1$



Example 4:

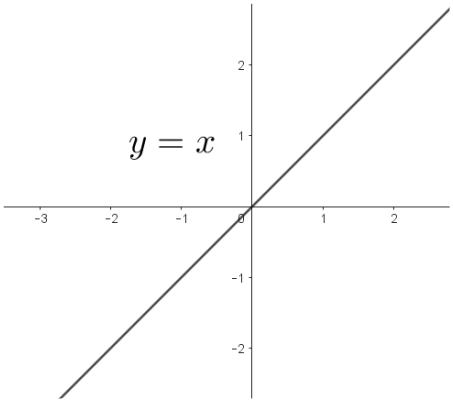
Find the x and y intercepts of $y = x^2 - 6x + 8$.

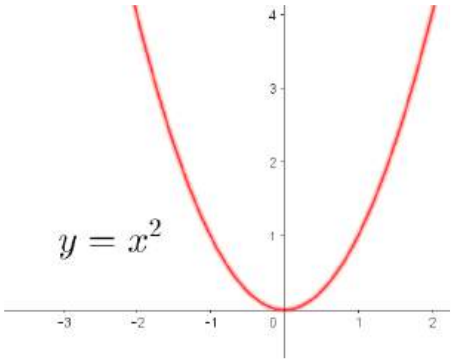
Solution:

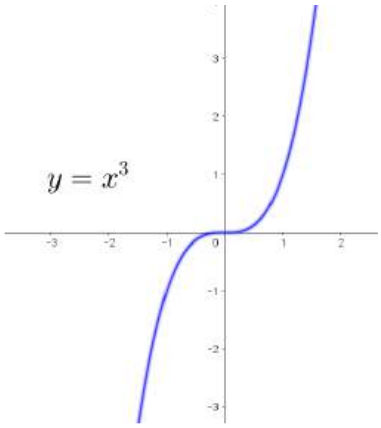
x-intercept when $y = 0$: $x^2 - 6x + 8 = 0 \rightarrow (x-2)(x-4) = 0 \rightarrow x = 2, x = 4$

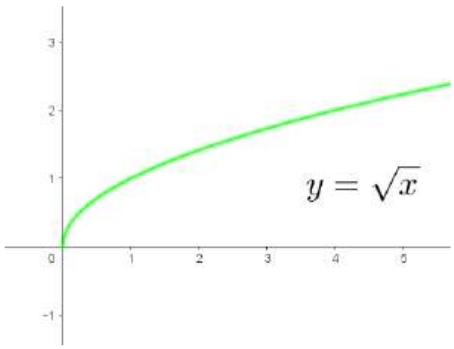
y-intercept when $x = 0$: $y = (0)^2 - 6(0) + 8 = 8$

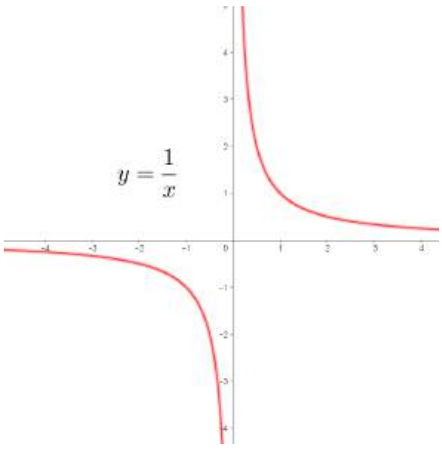
Graphs of fundamental Functions

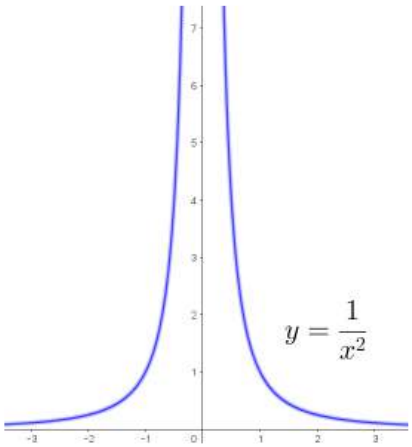












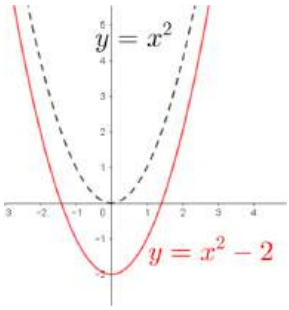
Tranformations

Vertical Shift:
Suppose $c > 0$:
to graph $y = f(x) + c$, shift the graph of $y = f(x)$ **up** c units
to graph $y = f(x) - c$, shift the graph of $y = f(x)$ **down** c units

Example 5:

Let $f(x) = x^2 - 2$.

The entire graph moves **down by 2 units**.

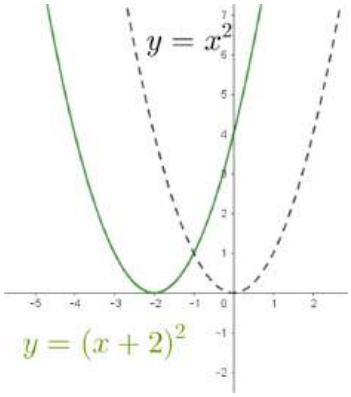


Horizontal Shift:
Suppose $c > 0$:
to graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the **left** c units
to graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the **right** c units

Example 6:

Let $f(x) = (x + 2)^2$.

The entire graph moves **by 2 units to the left**.



REFLECTION:

to graph $y=-f(x)$, reflect the graph of $y=f(x)$ in the *x axis*

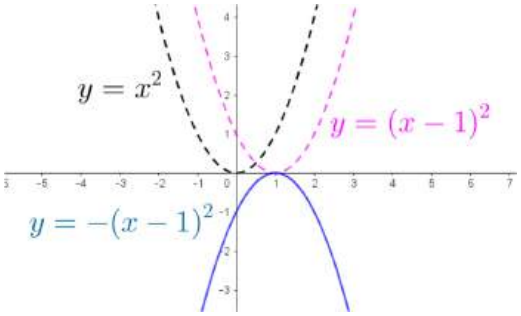
to graph $y=f(-x)$, reflect the graph of $y=f(x)$ in the *y axis*

Example 7:

Let $f(x)=-(x-1)^2$.

First: the graph $y=x^2$ moves **1 unit to the right**.

Then is reflected in the ***x*-axis**

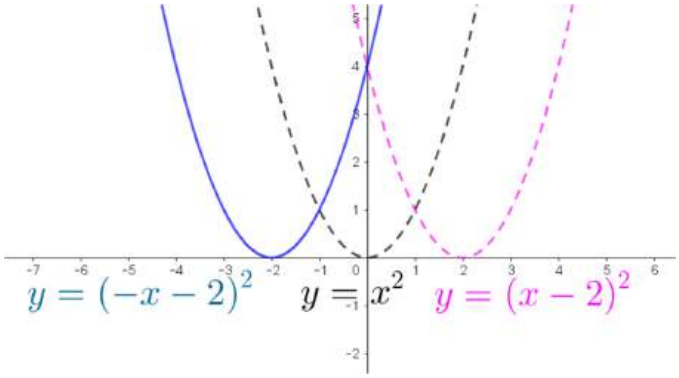


Example 8:

Let $f(x)=-(x-2)^2$.

First: the graph $y=x^2$ moves 2 **units to the right**.

Then is reflected in the ***y*-axis**



Vertical Stretching and Shrinking:

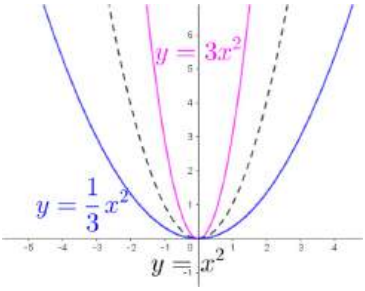
To graph $y=c\,f(x)$:

If **c > 1, stretch** the graph of $y=f(x)$, vertically by a factor of *c*.

If **c < 1, shrink** the graph of $y=f(x)$, vertically by a factor of *c*.

Example 9:

Vertical Stretching and Shrinking of $f(x)=x^2, f(x)=3x^2, f(x)=\frac{1}{3}x^2$.



Horizontal Stretching and Shrinking:

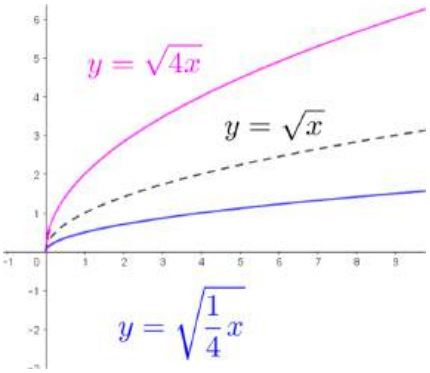
To graph $y=f(cx)$:

If **$c > 1$, shrink** the graph of $y=f(x)$ horizontally by a factor of c.

If **$c < 1$, stretch** the graph of $y=f(x)$ horizontally by a factor of c.

Example 10:

Horizontal Stretching and Shrinking of $f(x)=\sqrt{x}, f(x)=\sqrt{4x}, f(x)=\sqrt{\frac{1}{4}x}$.



NOTE:

$f(x)=\sqrt{4x}$ is algebraically the same as $f(x)=2\sqrt{x}$ (Shrinking it horizontally by 4 looks exactly the same as stretching it vertically by 2).

$f(x)=\sqrt{\frac{1}{4}x}$ is algebraically the same as $f(x)=\frac{1}{2}\sqrt{x}$ (Stretching it horizontally by 4 looks exactly the same as shrinking vertically by a half).

Piecewise Functions

A piecewise is a function that is split into "*pieces*," where each piece follows a different formula depending on the value of the input x. Instead of using one single equation for the entire graph, a piecewise function says: *"If x is in this region, use formula A. If x is in that region, use formula B."*

Example 10:

Let $f(x)=\begin{cases} x+2, & \text{when } x<0 \\ x^2, & \text{when } x\geq 0 \end{cases}$

If you want to find $f(-3)$, since $-3<0$, you use the top formula: $f(-3)=-3+2=-1$

If you want to find $f(2)$, since $2\geq 0$, you use the bottom formula: $f(2)=(2)^2=4$

Example 11: Real-World Analogy

Think of a piecewise function like a tax bracket or movie ticket pricing:

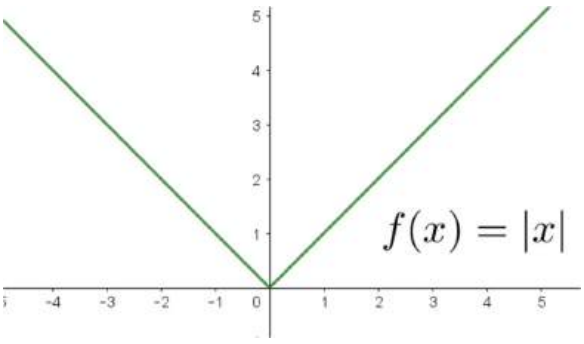
- If you are **under 12**, tickets cost 5 €.
- If you are **between 12 and 64**, tickets cost 12 €.
- If you are **65 or older**, tickets cost 8 €.

The price changes completely based on the "domain" (your age), but it is still all part of one single pricing system.

Definition Absolute Value
the absolute value of a real number x, written as $|x|$, represents its distance from zero on the number line. We define the absolute value as a piecewise function:

$$|x|=\begin{cases} x, & \text{when } x\geq 0 \\ -x, & \text{when } x<0 \end{cases}$$

The graph of the basic absolute value function is a perfect **V-shape** with its vertex at the origin (0,0).



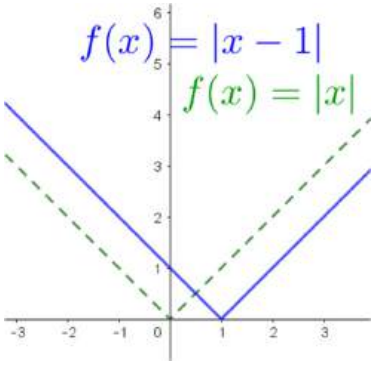
Example 12:

Let $f(x)=|x-1|$, define $f(x)$ then using transformation graph it.

Solution:

Definition: $f(x)=|x-1|=\begin{cases} x-1, & \text{when } x\geq 1 \\ -(x-1), & \text{when } x<1 \end{cases}$

The entire graph moves **by 1 unit to the right**.



Example 13:

Let $f(x)=-|x+2|$.

(i) Define $f(x)$

(ii) Specify a sequence of transformation to perform on the graph of $y=|x|$ to obtain the graph of $f(x)=-|x+2|$

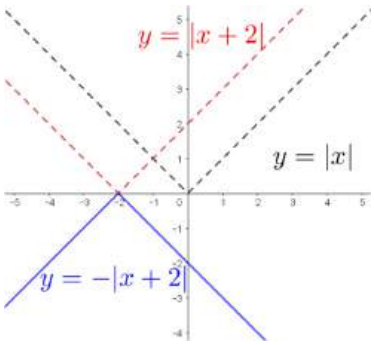
(iii) Graph $f(x)$

Solution:

(i) $f(x)=|x+2|=\begin{cases} x-1, & \text{when } x\geq -2 \\ -(x+2), & \text{when } x<-2 \end{cases}$

(ii) First: move 2 units to the left; then reflect on the x-axis

(iii)



Example 14:

Let $f(x)=\begin{cases} x^2, & \text{when } x<1 \\ x-3, & \text{when } x\geq 1 \end{cases}$.

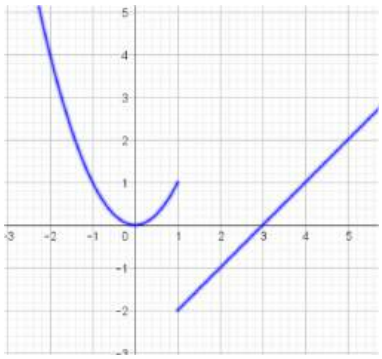
(i) Find $f(2)$ and $f(-3)$

(ii) Graph $f(x)$

Solution:

(i) $f(2)=(2)^2=4$ and $f(-3)=(-3)-3=-6$

(ii)



Continuity (simple definition)

A function is continuous if you can draw its entire graph from left to right without lifting your pencil from the paper. In other words, a continuous graph has no breaks, jumps, holes, or vertical gaps. It is a single, unbroken curve or line.

A piecewise function is continuous when all pieces agree on the exact same value at the *transition point*, the curves glue together perfectly.

Example 15:

The function of example 14 is not continuous.
The transition point is 1.

Left piece: $f(1)=(1)^2=1$

Right piece: $f(1)=(1)-3=-2$

The function is continuous to the left of 1; the function is continuous to the right of 1; but is not continuous at 1.

Example 16:

Determine if $f(x)=\begin{cases} x^2+1, & \text{when } x<2 \\ 2x+1, & \text{when } x\geq 2 \end{cases}$ is continuous everywhere.

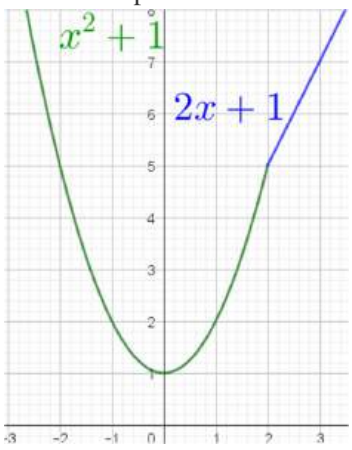
Solution:

The function is continuous to the left of 2; the function is continuous to the right of 2; we need to check the values at the transition point 2.

Left piece: $f(2)=(2)^2+1=5$

Right piece: $f(2)=2(2)+1=5$

Since both pieces have the same value at 2 the function is continuous everywhere.



Example 17:

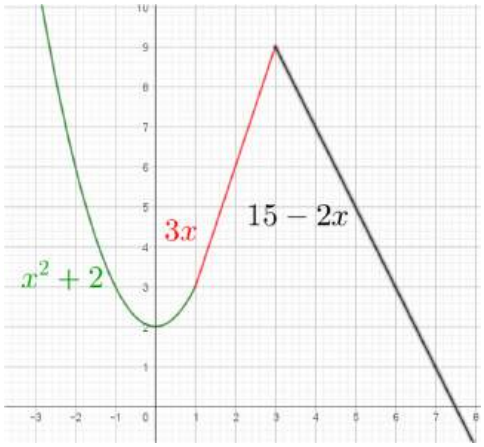
Determine the value of a and b so $f(x)=\begin{cases} x^2+a, & \text{when } x<1 \\ 3x, & \text{when } 1\le x\le 3 \\ b-2x & \text{when } x>3 \end{cases}$ is continuous everywhere.

Solution:

In order to be continuous everywhere we need to have

$x^2+a=3x$ when $x=1 \quad \rightarrow 1+a=3 \rightarrow a=2$

$3x=b-2x$ when $x=3 \rightarrow 3(3)=b-2(3) \rightarrow 9=b-6 \rightarrow b=15$



Symmetry-Even and Odd Functions

Recognizing symmetry allows mathematicians to simplify complex integration and differentiation problems in calculus. Functions typically demonstrate one of two clean variants of structural symmetry:

Let *f* be a function:

Point Transformation: *f* is ***even*** if $f(x)=f(-x)$.The graph of an even function is symmetric with respect to the *y*-axis. If a point *(x, y)* is on the graph, then the point *(-x, y)* must also be on the graph.

Point Transformation: *f* is ***odd*** if $-f(x)=f(-x)$.The graph of an even function is symmetric with respect to the origin. If a point *(x, y)* is on the graph, then the point *(-x, -y)* must also be on the graph.

Example 18:

Determine whether $f(x)=2x^2+x-3$ is even, odd or neither.

Solution:

Even $f(x)=2x^2+x-3;\ f(-x)=2(-x)^2+(-x)-3=2x^2-x-3\rightarrow f(x)\neq f(-x)$

odd $-f(x)=-(2x^2+x-3)=-2x^2-x+3;\rightarrow -f(x)\neq f(-x)$

Therefore $f(x)$ is neither.

Example 19:

Determine whether $f(x)=4x^3-x$ is even, odd or neither.

Solution:

Even $f(x)=4x^3-x;\ f(-x)=4(-x)^3-(-x)=-4x^3+x\rightarrow f(x)\neq f(-x)$

odd $-f(x)=-(4x^3-x)=-4x^3+x\rightarrow -f(x)\neq f(-x)$

Therefore $f(x)$ is an odd function.

Increasing and Decreasing Functions

Increasing and decreasing functions are defined by looking at what happens to the height of the graph (*y*-value) as you move from left to right along the *x*-axis.

$f(x)$ is increasing on an interval A if taken any two *x*-values where $x_1<x_2$ then $f(x_1)<f(x_2)$

$f(x)$ is decreasing on an interval A if taken any two *x*-values where $x_1<x_2$ then $f(x_1)>f(x_2)$

Example 20:

Determine whether $f(x)=3x-5$ is increasing or decreasing.

Solution:

Step 1: take two real x_1 and x_2 and where $x_1 < x_2$

Step 2: Make both sides look like the given function. In this case multiply both sides by 3 to have: $3x_1 < 3x_2$

Step 3: Subtract 5 from both sides: $3x_1 - 5 < 3x_2 - 5$

Step 4: let $f(x_1) = 3x_1 - 5$ and $f(x_2) = 3x_2 - 5$. It follows that $f(x_1) < f(x_2)$.

Therefore $f(x) = 3x - 5$ is an increasing function.

Example 21:

Determine whether $f(x) = \frac{1}{x}$ is increasing or decreasing in $(0, \infty)$

Solution:

Step 1: take two real x_1 and x_2 and where $x_1 < x_2$ and both are positive.

Step 2: Make both sides look like the given function: $\frac{1}{x_1}$ and $\frac{1}{x_2}$

Step 3: Taking the reciprocal of two positive numbers flips the inequality sign: $2 < 4$ but $\frac{1}{2} > \frac{1}{4}$. So we have: $\frac{1}{x_1} > \frac{1}{x_2}$

Step 4: let $f(x_1) = \frac{1}{x_1}$ and $f(x_2) = \frac{1}{x_2}$. It follows that $f(x_1) > f(x_2)$.

Therefore $f(x) = \frac{1}{x}$ is a decreasing function in $(0, \infty)$.

Miscellaneous Problems

When manufacturing an item we need to consider:
Fixed cost: for designing the product, establishing a factory, workers and so on.

Marginal cost: the cost of producing an additional item .

If the cost of an item can be described by a linear function $f(x) = mx + q$ then q is the fixed cost and m is the marginal cost.

Example 22:

The cost of producing 10 radios is 220 euro while the cost of producing 7 radios is 184 euro. If the cost C can be described by the equation $C = mx + q$, find:

(i) the marginal cost

(ii) the function $C(x) = mx + q$

Solution:

(i) We have two points to consider (10, 220) and (7, 184)
So the marginal cost is the slope m

$$\text{marginal cost} = m = \frac{220-184}{10-7} = \frac{36}{3} = 12$$

(ii) using the point-slope form we have

$$C - 220 = 12(x - 10)$$

$$C(x) = 12x + 100$$

A company can make a profit only if the revenue *R* it receives from its customers exceeds the cost *C* of producing its good and services. The point at which revenue *R* just equals cost *C* is called *break-even point*.

Example 23:

A firm producing cat food finds that the total cost *C* of producing *x* units is given by *C = 20x + 100*. The revenue *R* from selling *x* units is given by the product of the price per unit and the number of units sold. If the food sells for £ 24 per unit, then the revenue *R* from selling *x* units is *R = 24x*. Find the break-even point.

Solution:

The break-even point is given when *R = C* therefore when

24x = 20x +100

4x = 100

x = 25

If the company produces more than 25 units, it makes a profit.

Example 24:

Find the value of *a* so the function $f(x) = x + \sqrt{2x - a}$ goes through point (6, 9)

Solution:

point (6, 9) means that when x = 6 then $f(6) = 9$, so we have:

$$9 = 6 + \sqrt{2(6) - a}$$

$$9 - 6 = \sqrt{12 - a}$$

$$(3)^2 = \left(\sqrt{12 - a}\right)^2$$

$$9 = 12 - a$$

$$a = 3$$

Example 25:

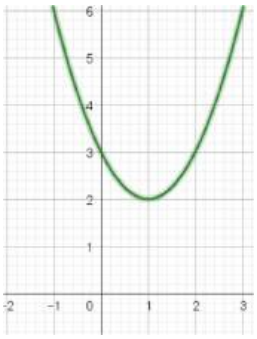
Given $f(x) = 5x - 4$ and $g(x) = \frac{2}{3}x$, find the value(s) so $f(x) = g(x)$.

Solution:

We need to have: $5x - 4 = \frac{2}{3}x \rightarrow 3(5x - 4) = \left(\frac{2}{3}x\right)3 \rightarrow 15x - 12 = 2x \rightarrow 13x = 12 \rightarrow x = \frac{12}{13}$

Example 26:

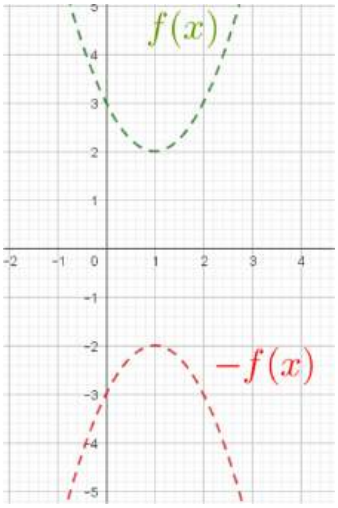
Below is the graph of $f(x)$. Graph $-f(x)+1$



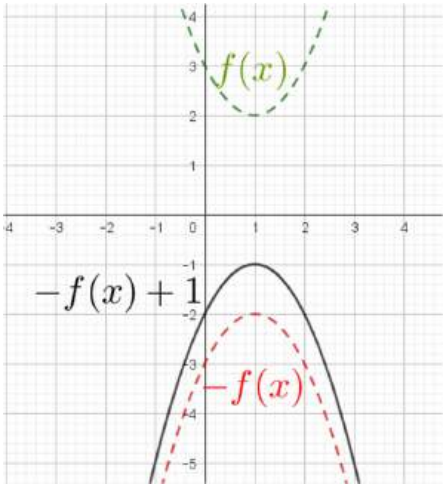
Solution:

Starting from $f(x)$:

Negative sign: reflect across the x-axis



From $-f(x)$ we move up by 1 unit to have



Example 27:

A rectangle has an area of 16m^2. Express the perimeter of the rectangle as a function of the length of one of its sides. State its domain.

Solution:

Step 1: Define the Variables

Let x be the length of one side of the rectangle (in meters), and let y be the length of the adjacent side (in meters).

Step 2: Use the Area to Relate the Sides

The area A of a rectangle is given by the formula: $A = xy = 16$.

We can express y in terms of x: $y = \frac{16}{x}$

Step 3: Express the Perimeter as a Function of x

The perimeter P of a rectangle is the sum of all its sides: $P = 2x + 2y$

Substitute the expression for y into the perimeter formula: $P = 2x + 2\left(\frac{16}{x}\right) = 2x + \frac{32}{x}$

Step 4: State the Domain

The domain of a function represents all the possible valid input values for x. Since x represents a physical length of a side of a rectangle, it must be strictly greater than zero ($x > 0$). There is no upper limit on how long one side can be (as the side x gets larger, the adjacent side y gets smaller to keep the area at 16m^2.

Therefore, the domain of the function is $(0,\infty)$.

Example 28:

A taxi company charges two euros for the first km (or part of a kilometer) and 20 cents for each succeeding tenth of a km (or part). Express the cost C (in euro) of a ride as a function of the distance x travelled (in km) for $0 < x < 2$.

Solution:

We can analyze the pricing structure into two main parts:

The First Kilometer: The base fare is 2.00 euros for any distance up to and including the first kilometer, then $C(x) = 2 \text{ when } 0 < x \leq 1$

Succeeding Tenths of a Kilometer:

For any distance beyond 1km, each additional 0.1km (or any fraction/part) costs an extra 0.20 euros. The additional distance is $x - 1$, and the number of tenths of a kilometer in this additional distance is $10(x - 1)$.

Then $C(x) = 0.20\left[10(x-1)\right] = 2(x-1) \text{ when } 1 < x < 2$.

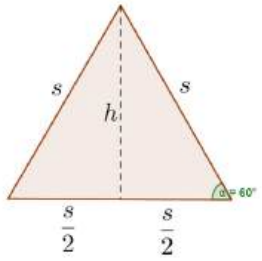
Total Cost: $C(x) = \begin{cases} 2 & \text{when } 0 < x \leq 1 \\ 2 + 2(x-1) & \text{when } 1 < x < 2 \end{cases}$

Example 29:

Express the area of an equilateral triangle as a function of the length of a side. State its domain.

Solution:

Let s be the length of each side of the triangle. By dropping a perpendicular line (h) from the top vertex to the base, we split the triangle into two congruent $30^\circ - 60^\circ - 90^\circ$ right triangles. The base of one the right triangles is half of the total side $\frac{s}{2}$, other side is h and the hypotenuse is s, as shown in the figure.



Using the Pythagorean theorem we have:

$$h^2 + \left(\frac{s}{2}\right)^2 = s^2 \quad \rightarrow \quad h^2 = s^2 - \left(\frac{s}{2}\right)^2 = s^2 - \frac{s^2}{4} = \frac{4s^2 - s^2}{4} = \frac{3s^2}{4}$$

Take the square root from both sides to get h:

$$\sqrt{h^2} = \sqrt{\frac{3s^2}{4}} \quad \rightarrow \quad h = \frac{s}{2}\sqrt{3}$$

Formula for the area of a triangle is $Area = \frac{base \times height}{2} = \frac{s \times h}{2}$

Substituting h we have: $Area = \frac{s \times h}{2} = \frac{s\left(\frac{s}{2}\sqrt{3}\right)}{2} = \frac{s^2\sqrt{3}}{4}$

Example 30:

Given $f(x) = ax^2 - b$ such that $f(0) = -2$ and $f(1) = 4$, determine the value of a and b.

Solution:

From the given information we have:

1. $-2 = a(0)^2 - b$ and 2. $4 = a(1)^2 - b$

From equation 1 we have: $-2 = -b \rightarrow b = 2$

Substitute in equation 2: $4 = a(1)^2 - 2 \rightarrow a = 6$

Example 31:

Prove that any function can be written as the sum of an even and an odd function.

Solution:

We need to find two functions $E(x)$ (even) and $O(x)$ (odd) such that $f(x) = E(x) + O(x)$

Let be :

equation 1: $f(x)=E(x)+O(x)$ and substituting x with $-x$ we have

Equation 2: $f(-x)=E(-x)+O(-x)$. Using definition of even and odd we have:

Equation 2: $f(-x)=E(x)-O(x)$

Add equation 1 and equation 2 to solve for $E(x)$:

$$f(x)+f(-x)=2E(x)$$

$$E(x)=\frac{f(x)+f(-x)}{2}$$

Subtract equation 2 from equation 1 to solve for $O(x)$:

$$f(x)-f(-x)=O(x)+O(x)$$

$$O(x)=\frac{f(x)-f(-x)}{2}$$

Verify that $E(x)$ is even: $E(-x)=\frac{f(-x)+f(-(-x))}{2}=\frac{f(-x)+f(x)}{2}=E(x)$

Verify that $O(x)$ is odd: $O(-x)=\frac{f(-x)-f(-(-x))}{2}=\frac{f(-x)-f(x)}{2}=\frac{-[f(x)-f(-x)]}{2}=-O(x)$

Verify the sum:

$$E(x)+O(x)=\frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}=\frac{2f(x)}{2}=f(x)$$

Therefore any function $f(x)$ can uniquely be decomposed into the sum of an even function and an odd function.

Practice Problems

1. The marginal cost for a chair is \$12 per chair, while the cost to produce 150 chairs is \$2100. Find the linear cost equation $C(x)$. $[C(x)=12x+300]$
2. Neon Nuance Tech manufactures specialized ergonomic mechanical keyboards. The company's financial analysts have modeled their monthly cost and revenue structures based on the number of keyboards produced. The total monthly cost *C(x)* (in dollars) to produce *x* keyboards is given by the quadratic function: $C(x)=x^2+40x+1200$. The total monthly revenue R(x) (in dollars) generated from selling *x* keyboards is given by the linear function: $R(x)=50x$.
   Determine the number of keyboards Neon Nuance Tech must produce and sell each month to break even. $[x=40]$
3. Find *a* so that $f(x)=ax+5$ passes through $(2,13)$ $[a=4]$
4. A newspaper ad costs \$14 plus 25cents for each word after the first ten.
   (i) write an expression for the cost *C* for an ad
   (ii) John paid \$22.50 for his ad. How many words were in it? $[C=25x+1400;44]$
5. Tickets for the homecoming dance cost \$20 for a single ticket or \$35 for a couple. Ticket sales totaled \$3625 and 128 people attended. How many tickets of each type were sold? $[57,71]$
6. Given $f(x)=3x-4$ find *x* such that $f(x)=11$ $[x=5]$
7. A right triangle with sides *a* and *b* with hypotenuse of 5 cm and area of 6 cm^2. Express the perimeter as a function of side *a* and state the domain. $\left[P=\dfrac{a^2+12a+5}{a}\right]$
8. Find the value of $f(2)+f(-1)$ for $f(x)=x^2-3x+2$ $[sum=6]$
9. A rectangle's length is twice its width. Express the area as a function of the perimeter. State the domain. $\left[Area=\dfrac{P^2}{18}\right]$
10. Given $f(2x+1)=4x^2-6x+3$ find $f(x)$ $[f(x)=x^2-5x+7]$
11. Find the intercepts of $f(x)=-3x+9$. $[(3,0),(0,-6)]$
12. Determine the intercepts of $f(x)=x^2-4$ $[(\pm 2,0),(0,-4)]$
13. Find the intercepts of $f(x)=|x-3|-2$ $[(1,0),(5,0),(0,1)]$
14. Given $f(x)=ax^2+bx+3$ with *x*- intercepts $x=\dfrac{1}{2}$ and $x=3$,
    find the value of *a* and *b* $[a=2,b=-7]$
15. Without graphing, determine which fundamental function has a graph that is symmetric about the origin and passes through (1,1), (2,8), (-1,-1). $[f(x)=x^3]$

16. Start with $f(x)=x^2$ .Write the equation for a shift 3 units up and 2 units left.

$[f(x)=(x+2)^2+3]$

17. If $f(x)=\sqrt{x}$ graph $g(x)=\sqrt{x-4}$ and describe the transformation. [shift 4 units right]

18. Given $f(x)=x^3$ graph $h(x)=1-x^3$ and list the transformations in order.

[reflect on x-axis, shift up 1]

19. Start with $f(x)=|x|$ Write the equation for a reflection over the x-axis followed by a vertical stretch by factor 2. $[g(x)=-2|x|]$

20. The graph of $f(x)=x^2$ is transformed into $g(x)=3(x-1)^2+4$ Describe all transformations.

[shift right 1, vertical stretch by 3, shift up 4]

21. Suppose $g(x)=-2f(3x)+1$ and $f(x)=\sqrt{x}$ Find $g(x)$ and describe the transformations in terms of stretching/shrinking and reflecting.

$[g(x)=-2\sqrt{3x}+1;$ shrink horizontal by $\frac{1}{3}$, vertical stretch by 2, reflect over x-axis, shift up 1$]$

22. Write a piecewise function for a parking garage that charges €2 for the first hour and €1 for each additional hour or part of hour.

$$\left[C(h)=\begin{cases}2 & \text{when } 0<h\le 1\\ 2+(h+1) & \text{when } h>1\end{cases}\right]$$

23. Given $f(x)=\begin{cases}3x+a & \text{when } x<1\\ x^2+b & \text{when } x\ge 1\end{cases}$. Find a and b so that $f(1)=4$ and $f(0)=1$ $[a=1,b=3]$

24. Design a piecewise function that models a cell phone plan: €15/month for up to 500 minutes, €0.05 per additional minute. Write the expression for $x \geq 0$.

$$\left[C(x)=\begin{cases}15 & \text{when } 0<x\le 500\\ 15+0.05(x-500) & \text{when } x>500\end{cases}\right]$$

25. Is $f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & \text{when } x\neq 1\\ 2 & \text{when } x=1\end{cases}$ continuous at $x = 1$? $[yes; f(1)=2]$

26. Find the value of a that makes $f(x)=\begin{cases}ax & \text{when } x\le 2\\ x^2 & \text{when } x>2\end{cases}$ continuous at $x = 2$. $[a=2]$

27. Find the value of a and b that makes $f(x)=\begin{cases}x^2 & \text{when } x<1\\ ax+b & \text{when } 1\le x<3\\ 2x-4 & \text{when } x>3\end{cases}$

continuous everywhere. $[a=2,b=-1]$

28. Determine if $f(x)=x^4-2x^2+1$ is even, odd, or neither. [*even*]

29. Is $f(x)=x^3-x$ even, odd, or neither? [*odd*]

30. If $f(x)$ is even and $f(3)=7$ what is $f(-3)$? $[f(-3)=7]$

31. If $f(x)$ is odd and $f(4)=11$ what is $f(-4)$? $\left[f(-4)=-11\right]$

32. Show that $f(x)=|x|$ is even.

33. Show that the sum of two odd functions is an odd function.

34. Show that the sum of two even functions is an even function.

35. On what interval is $f(x)=x^2-4x+3$ decreasing? $(-\infty,2)$

36. Given $f(x)=\sqrt{x}$ on $(0,\infty)$ show it is increasing by showing $f(x_2)>f(x_1)$ whenever $x_2>x_1>0$

37. Let $f(x)=\begin{cases} ax^2+b & \text{when } x<2 \\ 5x-3 & \text{when } x\geq 2 \end{cases}$, if f is continuous at $x=2$ and $f(0)$,
find a and b $\left[a=1,b=3\right]$

38. Let $f(x)=\begin{cases} x+2 & \text{when } x<0 \\ ax+b & \text{when } 0\leq x\leq 2 \\ 3x-4 & \text{when } x>2 \end{cases}$, find a and b so f is continuous for all x. $\left[a=0,b=2\right]$

1.3- Operations on Functions

Topic: Average Rate of Change, Composition, Inverses

Review

1. Let $f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & \text{when } x\neq 2\\ k & \text{when } x=2\end{cases}$. Find the value of k that makes $f(x)$ continuous at $x=$ $[k=4]$

2. Let $f(x)=\begin{cases}|x-a|+x & \text{when } x\leq 1\\ 2+a & \text{when } x>1\end{cases}$. Find the value of a that makes $f(x)$ continuous at x $[a=0]$

3. Given $f(3x-2)=2x^2+x-1$ find $f(x)$ $\left[f(x)=\dfrac{2x^2+11x+5}{9}\right]$

4. Given $f(2x+5)=x^2-4x+6$ find $f(x)$ $\left[f(x)=\dfrac{x^2-18x+89}{4}\right]$

Average Rate Of Change

The average rate of change finds how fast a function is changing with respect to something else changing. It is simply the process of calculating the rate at which the output *(y-values)* changes compared to its input *(x-values)*.

How to Find the Average Rate of Change

The average rate of change of a function corresponds to the slope of the line, which connects two endpoints of a given interval (known as the secant line). Here is the average rate of change formula:

$$avg=\frac{change\,in\ y}{change\,in\ x}=\frac{y_2-y_1}{x_2-x_1}$$

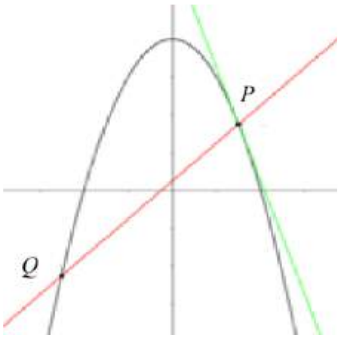
Using function notation with points $(a,f(a))$ and $(b,f(b))$ we have: $avg=\dfrac{f(b)-f(a)}{b-a}$

For a linear function, the average rate of change (slope) is constant, but for a non-linear function, the average rate of change is not constant.

Average vs Instantaneous Rate Of Change

While both are used to find the slope, the average rate of change calculates the slope of the *secant line* using the slope formula from algebra. The instantaneous rate of change calculates the slope of the *tangent line* using calculus.

The red secant line represents the average rate of change between points P and Q, and the green tangent line designates the instantaneous rate of change at point P.



The Concept of Average Value and Accumulation

The *Area* under a curve $y = f(x)$ over an interval $[a,b]$ represents the total accumulation of all the infinite *y*-values (heights) added together across that horizontal width.
Because a curve's height changes constantly, we can use the *Average Value of the Function* (denoted f_{avg}, which represents the average height or average *y*-value) to simplify the geometry.

The Formulas

Because the area under the curve is equivalent to the area of a transformed rectangle $(\text{Area} = \text{width} \times \text{height})$ we can express this relationship in two ways depending on what information you are given:

To find the Area (when you know f_{avg} and the interval): $\text{Area} = (b-a) \times f_{avg}$

To find f_{avg} (when you know the area and the interval): $f_{avg} = \dfrac{\text{Area}}{(b-a)}$

Where:

$(b-a)$ is the *Interval Width* (the change in *x*).

f_{avg} is the *Average Value* of the function (the average *y*-value).

Area is the *Total Accumulation* under the curve.

Example 1:

Given $f(x) = 2x^2 - x + 3$, find the average rate of change between $x = -1$ and $x = 4$.

Solution:

Using the formula $avg = \dfrac{f(b) - f(a)}{b-a}$ we have:

$$avg = \frac{f(4) - f(-1)}{4-(-1)} = \frac{\left[2(4)^2 - (4) + 3\right] - \left[2(-1)^2 - (-1) + 3\right]}{4-(-1)} =$$

$$= \frac{[32-1]-[2+4]}{4+1} = \frac{25}{5} = 5$$

Example 2:

The area under the curve $f(x)=x^2$ on the interval $[1,b]$ is 21. The average rate of change over this interval is 5. Find f_{avg}.

Solution:

Using the formula $avg=\frac{f(b)-f(a)}{b-a}$ we have:

$$5=\frac{b^2-1}{b-1}\rightarrow 5b-5=b^2-1\rightarrow b^2-5b+4=0$$

Factorize:

$b^2-5b+4=0\rightarrow (b-4)(b-1)=0\rightarrow b=1,b=4$. Since 1 is already given value we have $b=4$

$f_{avg}=\frac{\text{Area}}{(b-a)}$ we have:

$$f_{avg}=\frac{21}{(4-1)}=7$$

Example 3:

The value of f_{avg} of the curve $f(x)=x^5+1$ on the interval $[a,2]$ is $\frac{9}{2}$. Find the area under the curve in that interval knowing that the average rate of change is $avg=11$.

Solution:

Using $avg=\frac{f(b)-f(a)}{b-a}$ we have:

$$11=\frac{\left[(2)^5+1\right]-[a+1]}{2-a}$$

$$22-11a=32-a$$

$$22-32=11a-a$$

$$a=-1$$

Using $\text{Area}=(b-a)\times f_{avg}$ we have $\text{Area}=(2-(-1))\times\frac{9}{2}=\frac{27}{2}$

Algebraic Operations on Functions

Just as ordinary numbers can be added, subtracted, multiplied, and divided, complete mathematical functions can be combined to produce entirely new algebraic structures. The core challenge here is tracking how the domain changes after structural combination.

Operation Type	Notation / Definition	Domain of the New Combined Function
Sum and Difference	$(f \pm g)(x) = f(x) \pm g(x)$	The intersection of domains: $\text{Domain}(f) \cap \text{Domain}(g)$
Product	$(f \times g)(x) = f(x) \times g(x)$	The intersection of domains: $\text{Domain}(f) \cap \text{Domain}(g)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Intersection $\text{Domain}(f) \cap \text{Domain}(g)$ EXCLUDING any x where $g(x) = 0$

Example 4:

Given $f(x) = 2x - 3$ and $g(x) = 3x + 4$, find $(f - g)(x)$

Solution:

$(f - g)(x) = f(x) - g(x) = (2x - 3) - (3x + 4) = -x - 7$

Example 5:

Given $f(a) = 6a - 1$ and $g(a) = 4a + 5$, find $(f + g)(3a)$

Solution:

$(f + g)(3a) = f(3a) + g(3a) = (6(3a) - 1) + (4(3a) + 5) = 30a + 4$ or

$(f + g)(a) = f(a) + g(a) = (6a - 1) + (4a + 5) = 10a + 4$ then

$(f + g)(3a) = 10(3a) + 4 = 30a + 4$

Example 6:

Given $f(x) = 2x - 7$ and $g(x) = x^2$, find $(f \times g)(a + 1)$

Solution:

$g(a + 1) = (a + 1)^2 = a^2 + 2a + 1$

$f(a + 1) = 2(a + 1) - 7 = 2a - 5$

$(f \times g)(a + 1) = f(a + 1) \times g(a + 1) = (2a - 5)(a^2 + 2a + 1) = 2a^3 - a^2$

Example 7:

Given $f(x)=x^3-x$ and $g(x)=x^2-2x-3$, find $\left(\frac{f}{g}\right)(x)$

Solution:

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x^3-x}{x^2-2x-3}=\frac{x\left(x^2-1\right)}{(x-3)(x+1)}=\frac{x(x-1)(x+1)}{(x-3)(x+1)}=\frac{x(x-1)}{(x-3)}$$

Example 8:

Given $f(x)=x^2-3x-18$ and $g(x)=x^2+2x-3$, find $\left(\frac{f}{g}\right)(a-1)$

Solution:

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x^2-3x-18}{x^2+2x-3}=\frac{(x-6)(x+3)}{(x+3)(x-1)}=\frac{(x-6)}{(x-1)} \text{ then } \left(\frac{f}{g}\right)(a-1)=\frac{((a-1)-6)}{((a-1)-1)}=\frac{a-7}{a-2}$$

Function Composition (Functions Inside Functions)

Composition is a sequential pipeline process. Instead of evaluating a number directly into an output, the output of the inner function is fed explicitly into the outer function as a brand new input value. This requires a strict 'nested chain' in calculus (vital for the Chain Rule).

Function Composition

The composite function $(f\circ g)(x)$ is read as " f of g of x " and is formally defined as:

$$(f\circ g)(x)=f(g(x))$$

The composition $f(g(x))$ means:

First apply g to x, **then** apply f to the result

The Domain Rule for Composition:

The domain of $(f\circ g)(x)$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Basically $f(g(x))$, is defined whenever both $f(x)$ and $g(x)$ are defined.

Order matters: in general $f(g(x))\neq g(f(x))$

Example 9:

Given $f(x)=x^2+1$ and $g(x)=3x-2$, find $(f\circ g)(x)$

Solution:

Inside f, where you see x, write $g(x)$ instead.

$$(f\circ g)(x)=f(g(x))=f(3x-2)=(3x-2)^2+1$$

Simplify:

$(f \circ g)(x) = (3x-2)^2 + 1 = 9x^2 - 12x + 5$

Example 10:

Given $f(x) = \sqrt{x}$ and $g(x) = x - 4$, find $(f \circ g)(x)$ and $(g \circ f)(x)$

Solution:

Inside *f*, where you see *x*, write $g(x)$ instead.

$(f \circ g)(x) = f(g(x)) = f(x-4) = \sqrt{x-4}$

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} - 4$

$f(g(x)) \neq g(f(x))$

Example 11:

Given $f(x) = \dfrac{2x}{x-3}$ and $g(x) = 5x + 1$, find $(f \circ g)(x)$ and its domain

Solution:

$(f \circ g)(x) = f(g(x)) = f(5x+1) = \dfrac{2(5x+1)}{(5x+1)-3} = \dfrac{10x+2}{5x-2}$

Domain: denominator $5x - 2 \neq 0 \rightarrow x \neq \dfrac{2}{5}$

Example 12:

Given $f(x) = \sqrt{x-2}$ and $g(x) = x^2 - 4x + 5$, find $(f \circ g)(x)$ and its domain

Solution:

$(f \circ g)(x) = f(g(x)) = f(x^2 - 4x + 5) = \sqrt{(x^2 - 4x + 5) - 2} = \sqrt{x^2 - 4x + 3} = \sqrt{(x-1)(x-3)}$

Domain: inside square root $(x-1)(x-3) \geq 0$

By testing one point in each interval:

Summary table (sign analysis)

Interval	Sign of $(x-1)$	Sign of $(x-3)$	Sign of product
$(-\infty, 1)$	–	–	+
$(1, 3)$	+	–	–
$(3, \infty)$	+	+	+

Checking some restrictions:

$g(x)$ is defined for all real x.

However $f(x)$ tequires to be greater than 2, therefore $g(x) \geq 2$. So we need to have

$x^2 - 4x + 5 \geq 2 \rightarrow x^2 - 4x + 3 \geq 0$, which is the inequality we already solved.

Therefore domain: $(-\infty, 1) \cup (3, \infty)$.

One-to-One Functions

A function with domain *D* is called a *one-to-one* function if no two elements of *D* have the same image: $f(a) \neq f(b)$ whenever $a \neq b$ or equivalently if $f(a) = f(b)$ then $a = b$

HORIZONTAL LINE TEST

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 13:

Show that all linear functions $f(x) = mx + q$ are one-to-one.

Solution:

Assume $f(a) = f(b)$ then

$ma + q = mb + q$ subtract q from both sides $\rightarrow$ $ma = mb$ divide by m

$a = b$

Since $f(a) = f(b)$ implies $a = b$ the functions are one-to-one

Alternatively if you graph $f(x) = mx + q$ (a non-vertical straight line with slope *m*), any horizontal line intersects it **at most once**. That also proves it's one-to-one, but the algebraic proof above is more rigorous.

Example 14:

Show that $f(x) = x^2 + 2$ is not one-to-one function

Solution:

To prove that is not one-to-one you need a counterexample:

$f(3) = (3)^2 + 2 = 11$, $f(-3) = (-3)^2 + 2 = 11$ but $3 \neq -3$. It fails the condition.

Example 15:

Show that $f(x) = \dfrac{2x+1}{x-3}$ is one-to-one function on its domain.

Solution:

First, find the domain: $x - 3 \neq 0 \rightarrow x \neq 3$. The domain is $(-\infty, 3) \cup (3, \infty)$.

For any a and b in this domain , if $f(a)=f(b)$ then $a=b$

Assume $f(a)=f(b)$ then $\frac{2a+1}{a-3}=\frac{2b+1}{b-3} \rightarrow (2a+1)(b-3)=(2b+1)(a-3)$

$2ab-6a+b-3=2ab-6b+a-3 \rightarrow -6a+b=-6b+a \rightarrow 6b+b=a+6a \rightarrow 7b=7a \rightarrow b=a$

Since $f(a)=f(b)$ implies $a=b$ the function is one-to-one.

Inverse of a Function

Let f be a **one-to-one** function with domain A and range B. The **inverse function** of f denoted f^{-1} (read "f *inverse*"), is defined as: $f(a)=b$ if and only if $f^{-1}(b)=a$ where

Domain of $f^{-1}=$ Range of $f=B$

Range of $f^{-1}=$ Domain of $f=A$

Properties

For all x in A: $f^{-1}(f(x))=x$

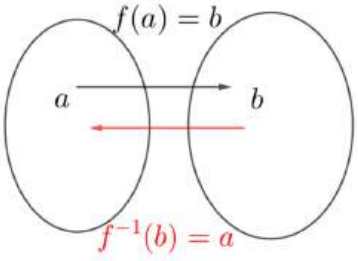
For all y in B: $f^{-1}(f(y))=y$

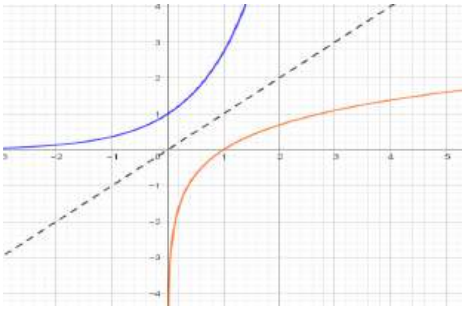
In other words: f^{-1} **undoes** what f does. Applying f then f^{-1} brings you back to the original input.

Two functions f and g are inverses if and only if $f(g(x))=x$ and $g(f(x))=x$ for all x in the respective domains.

Important notes

- A function must be **one-to-one** to have an inverse.
- If f is not one-to-one, you can restrict its domain to make it invertible.
- Notation f^{-1} means inverse function, **not** $\frac{1}{f}$ (reciprocal).
- The graphs of $f(x)$ and $f^{-1}(x)$ are symmetrical to the line $y = x$
- The graphs below show the relation between $f(x)$ and $f^{-1}(x)$.





Example 15:

Determinate if the given functions are inverses: $f(x)=\frac{4}{x-3}, g(x)=\frac{3}{x-4}$.

Solution:

Two functions f and g are inverses if and only if $f(g(x))=x$ and $g(f(x))=x$. If one condition fails f and g are not inverses.

$f(g(x))=f\left(\frac{3}{x-4}\right)=\frac{4}{\left(\frac{3}{x-4}\right)-3}=\frac{4}{\frac{3-3x+12}{x-4}}=\frac{4}{\frac{15-3x}{x-4}}=\frac{4(x-4)}{15-3x}=\frac{4x-16}{15-3x}$. Since $f(g(x))\neq x$ they are not inverses.

Example 16:

Determinate if the given functions are inverses: $f(x)=\frac{2x+1}{x-3}, g(x)=\frac{3x+1}{x-2}$

Solution:

We need to check if $f(g(x))=x$ and $g(f(x))=x$. If one condition fails f and g are not inverses.

$f(g(x))=f\left(\frac{3x+1}{x-2}\right)=\frac{2\left(\frac{3x+1}{x-2}\right)+1}{\left(\frac{3x+1}{x-2}\right)-3}=\frac{\left(\frac{6x+2}{x-2}\right)+1}{\frac{3x+1-3x+6}{x-2}}=\frac{6x+2+x-2}{x-2}\times\frac{x-2}{3x+1-3x+6}=\frac{7x}{7}=x$

$g(f(x))=g\left(\frac{2x+1}{x-3}\right)=\frac{3\left(\frac{2x+1}{x-3}\right)+1}{\left(\frac{2x+1}{x-3}\right)-2}=\frac{\left(\frac{6x+3+x-3}{x-3}\right)}{\left(\frac{2x+1-2x+6}{x-3}\right)}=\frac{\left(\frac{7x}{x-3}\right)}{\left(\frac{7}{x-3}\right)}=\frac{7x}{x-3}\times\frac{x-3}{7}=x$

Since $f(g(x))=x$ and $g(f(x))=x$ they are inverses.

How to find the Inverse of a One-to-One Function

1. Write $y=f(x)$
2. Interchange x and y.
3. Solve this equation for y.

The resulting equation is $y=f^{-1}(x)$

Example 17:

Let $f(x)=\frac{2x}{x-5}, x\neq 5$

(a) Find $f^{-1}(x)$

(b) Find the domain of $f^{-1}(x)$

Solution:

(a)

1. write $y=f(x)$: $y=\dfrac{2x}{x-5}$

2. interchange x and y: $x=\dfrac{2y}{y-5}$

3. solve for y: $x=\dfrac{2y}{y-5} \rightarrow x(y-5)=2y \rightarrow xy-5x=2y \rightarrow xy-2y=5x \rightarrow y(x-2)=5x \rightarrow y=\dfrac{5x}{x-2}$

(b) Therefore $f^{-1}(x)=\dfrac{5x}{x-2}$ with domain $x\neq 2$

Example 18:

Let $f(x)=\dfrac{ax+b}{cx+d}$, where $ad-bc\neq 0$ and $c\neq 0$. Find a formula for $f^{-1}(x)$ in terms of a ,b, c, d.

Solution:

$$y=\frac{ax+b}{cx+d}$$

$$x=\frac{ay+b}{cy+d}$$

$$x(cy+d)=ay+b$$

$$xcy+xd=ay+b$$

$$xcy-ay=b-xd$$

$$y(xc-a)=b-xd$$

$y=\dfrac{b-dx}{cx-a}$ then $f^{-1}(x)=\dfrac{b-dx}{cx-a}$, $x\neq\dfrac{a}{c}$

Miscellaneous Problems

Example 19:

The function $f(x)=2x^2+1$ has an average rate of change of 6 on $[1,b]$. Find $b>1$

Solution:

Given that $\dfrac{f(b)-f(1)}{b-1}=6$ then we have

$$\frac{[2b^2+1]-[2(1)^2+1]}{b-1}=6$$

$$\frac{2b^2-2}{b-1}=6 \rightarrow 2b^2-2=6(b-1) \rightarrow 2b^2-2=6b-6 \rightarrow 2b^2-6b+4=0$$

Simplify and factorize: $b^2-3b+2=0 \rightarrow (b-1)(b-2)=0 \rightarrow b=2$

Example 20:

The area under $f(x)=x^3+x$ on $[0,a]$ is 12. The average value of $f(x)$ on the same interval is m. Given that the average rate of change on $[0, a]$ is 6, find a and m.

Solution:

Average value: $m=\dfrac{Area}{a-0}=\dfrac{12}{a}$

Average rate of change: $\dfrac{f(a)-f(0)}{a-0}=\dfrac{a^3+a-0}{a}=a^2+1=6$

$a^2=5\rightarrow a=\sqrt{5}$ (positive)

Then: Average value: $m=\dfrac{Area}{a-0}=\dfrac{12}{\sqrt{5}}$

Example 21:

Given $f(x)=\dfrac{3x}{x-5}, g(x)=\dfrac{x-7}{x+6}$, find $(f\circ g)(2)$

Solution:

$(f\circ g)(2)$ means $f(g(2))$, so we find $g(2)$.

$$g(2)=\frac{(2)-7}{(2)+6}=-\frac{5}{8}$$

Then

$$(f\circ g)(2)=f(g(2))=f\left(-\frac{5}{8}\right)=\frac{3\left(-\frac{5}{8}\right)}{\left(-\frac{5}{8}\right)-5}=\frac{-\frac{15}{8}}{-\frac{45}{8}}=-\frac{15}{8}\left(-\frac{8}{45}\right)=\frac{1}{3}$$

Example 22:

Given $f(g(x))=20-6x$ and $g(x)=9-2x$, find $f(x+1)$

Solution:

Let $u=g(x)=9-2x$ then

$$x=\frac{u-9}{-2}=\frac{9-u}{2}$$

Substitute into $f(g(x))$:

$$f(u)=20-6\left(\frac{9-u}{2}\right)=20-27+3u=3u-7$$

Thus: $f(x)=3x-7$

Therefore $f(x+1)=3(x+1)-7=3x-4$

Example 23:

Find the inverse of $f(x)=\dfrac{2x+3}{x-4}$ then find the domain and range of both $f(x)$ and $f^{-1}(x)$

Solution:

$$y=\frac{2x+3}{x-4} \rightarrow x=\frac{2y+3}{y-4} \rightarrow x(y-4)=2y+3 \rightarrow xy-4x=2y+3 \rightarrow$$

$$\rightarrow xy-2y=3+4x \rightarrow y(x-2)=4x+3 \rightarrow y=\frac{4x+3}{x-2}$$

Therefore $f^{-1}(x)=\dfrac{4x+3}{x-2}$

Domain of $f(x): x\neq 4$; Range $f(x): y\neq 2$

Domain of $f^{-1}(x): x\neq 2$; Range $f^{-1}(x): y\neq 4$

Example 24:

Determine functions g and h so that $f(x)=g(h(x))$ given that $f(x)=\sqrt{x-5}$

Solution:

Choose h as the inner function: $h(x)=x-5$

Choose g as the outer function: $g(x)=\sqrt{x}$

Verify:

$f(x)=g(h(x))=g(x-5)=\sqrt{x-5}$

Example 25:

Determine functions g and h so that $f(x)=g(h(x))$ given that $f(x)=\dfrac{3}{(x^2+1)^5}$

Solution:

Let the inner function be $h(x)=(x^2+1)$ then

if $h(x)=(x^2+1)$ we have

$$f(x)=\frac{3}{[h(x)]^5}$$

Therefore the outer function is $g(x)=\dfrac{3}{x^5}$

Verify: $f(x)=g(h(x))=g(x^2+1)=\dfrac{3}{(x^2+1)^5}$

Example 26:

Given $f(x)=\sqrt{x+2}$ and $g(x)=ax+b$. Find *a* and *b* if given that $(f\circ g)(1)=2$ and $g(-1)=2$

Solution:

$(f\circ g)(1)=f(g(1))=2$; find $g(1)$: $g(1)=a(1)+b=a+b$ then $f(g(1))=f(a+b)=\sqrt{a+b+2}=2$

Solve for *a* + *b:*

$\sqrt{a+b+2}=2 \rightarrow a+b+2=4 \rightarrow a+b=2$ *equation* 1

Using $g(-1)=2 \rightarrow g(-1)=a(-1)+b=2 \rightarrow -a+b=2$ *equation* 2

Solve the system (*equation 1* + *equation 2*) we have: *a = 0* and *b = 2*

Practice Problems

1. Let $f(x)=x^2-3x+1$. If the average rate of change is 1 in $[a,6]$, find a. $[a=-2]$
2. Let $f(x)=2x^2+2x+3$. If the average rate of change is 2 in $[a,1]$, find a. $[a=-1]$
3. Let $f(x)=\frac{1}{x^2}$, find the average rate of change between $x=1$ and $x=1+h$ $\left[\frac{2+h}{(1+h)^2}\right]$
4. The area under $f(x)=x^2+2x$ on $[1,b]$ is $\frac{50}{3}$. The average value of f on the same interval is m.

   Given that the average rate of change on $[1,b]$ is 8, find b and m. $\left[b=5, m=\frac{25}{6}\right]$
5. The area under $f(x)=4x-x^2$ on $[a,4]$ is $\frac{22}{3}$. The average value of f on the same interval is m.

   Given that the average rate of change on $[a,4]$ is -2, find a and m. $\left[a=2, m=\frac{11}{3}\right]$
6. The area under $f(x)=x^2+1$ on $[p,q]$ is 24. The average value of f on the same interval is m. Given that the average rate of change on $[p,q]$ is 5, find p,q and m. $\left[p=\frac{1}{2}, q=\frac{9}{2}, m=6\right]$
7. Let $f(x)=\sqrt{x+3}$ and $g(x)=x^2-4$, find $(f+g)(x)$ and its domain.

   $\left[(f+g)(x)=\sqrt{x+3}+x^2-4;(-3,\infty)\right]$
8. Let $f(x)=x^2-2x+1$ and $g(x)=x^2-6x-3$, find $(f-g)(x)$ $\left[(f+-g)(x)=4x+4\right]$
9. Let $f(x)=x^2-1$ and $g(x)=x+1$, find $(f\times g)(x)$ and simplify $\left[(x-1)(x+1)^2\right]$
10. Let $f(x)=\frac{x}{x-2}$ and $g(x)=\frac{x+1}{x}$, find $\left(\frac{f}{g}\right)(x)$ and its domain $\left[\frac{x^2}{x^2-x-2}, x\neq -1,2\right]$
11. Let $f(x)=\sqrt{x-1}$ and $g(x)=\sqrt{x+3}$, find $(f\times g)(x)$ and its domain $\left[\sqrt{x^2+2x-3}, x\geq 1\right]$
12. Let $f(x)=\sqrt{x}+1$ and $g(x)=x-1$, find $\left(\frac{f}{g}\right)(x)$ and its domain $\left[\frac{1}{\sqrt{x}-1}, x\geq 0, x\neq 1\right]$
13. Let $f(x)=2x+4$ and $g(x)=x^2+5$, find $(f\circ g)(6)$ and its domain $[86]$
14. Let $f(x)=2x+5$, find $(f\circ f)(2x)$ and its domain $[8x+15]$
15. Let $f(x)=\frac{x^2-1}{3x^2+x+5}$ and $g(x)=x-2$, find $(f\circ g)(x)$ $\left[\frac{x^2-4x+5}{3x^2-11x+15}\right]$

16. Let $f(x)=\frac{2x}{x-3}$ and $g(x)=\frac{x+1}{x-2}$, find $(f\circ g)(x)$ and its domain $\left[\frac{2x+2}{7-2x}, x\neq 2,\frac{7}{2}\right]$

17. Let $f(x)=\sqrt{x-4}$ and $g(x)=x^2-6x+10$, find $(f\circ g)(x)$ and its domain $\left[\sqrt{x^2-6x+6}, x\neq 3\pm\sqrt{3}\right]$

18. Let $f(x)=\sqrt{x^2-2x+1}$ and $g(x)=x-1$, find $(f\circ g)(x)$ $\left[|x-2|\right]$

19. Let $f(x)=\frac{x}{x+1}$, $g(x)=x^2-1$ and $h(x)=\frac{1}{x}$, find $(f\circ g\circ h)(x)$ $\left[1-x^2\right]$

20. Given $(f\circ g)(x)=\frac{1}{x^2+4x+5}$ and $g(x)=x+2$, find $f(x)$ $\left[\frac{1}{x^2+1}\right]$

21. Determine functions g and h so that $f(x)=g(h(x))$ given $f(x)=\sqrt[3]{5x-2}$ $\left[g=\sqrt[3]{x}, h=5x-2\right]$

22. Determine functions g and h so that $f(x)=g(h(x))$ given $f(x)=\frac{4}{(x-5)^2+1}$ $\left[g=\frac{4}{x^2+1}, h=x-5\right]$

23. Given $f(g(x))=15-8x$ and $g(x)=5-2x$, find $f(x-2)$ $\left[4x-13\right]$

24. Given $f(g(x))=4x^2+4x+5$ and $g(x)=2x+1$, find $f(x-1)$ $\left[x^2-2x+5\right]$

25. Given $f(g(x))=\frac{2x}{x-3}$ and $g(x)=\frac{1}{x}$, find $f(x+2)$ $\left[-\frac{2}{3x+5}\right]$

26. Given $f(x)=\sqrt{x+5}$ and $g(x)=ax+b$, find a and b if $(f\circ g)(2)=4$ and $g(3)=7$ $\left[-\frac{2}{3x+5}\right]$

27. Let $f(x)=\sqrt{x+5}$ and $g(x)=ax+b$, find a and b if $(f\circ g)(2)=4$ and $g(3)=7$ $\left[a=-4, b=19\right]$

28. Let $f(x)=\sqrt{x-3}$ and $g(x)=ax+b$, find a and b if $(f\circ g)(4)=3$ and $g(0)=5$ $\left[a=\frac{7}{4}, b=5\right]$

29. Find the inverse $f(x)=\frac{\sqrt[3]{4x}-2}{2}$ $\left[2(x+1)^3\right]$

30. Find the inverse $f(x)=-\frac{3}{x+1}-2$ $\left[-\frac{3}{x+2}-1\right]$

31. The reciprocal function $f(x)=\frac{1}{x}$ is its own inverse. Find all real numbers a and b with $a\neq 0$ such that function $f(x)=\frac{a}{x+b}$ is its own inverse. $\left[a\neq 0, b=0\right]$

32. The reciprocal function $f(x)=\frac{1}{x}$ is its own inverse. Find all real numbers a and b with $a\neq 0$ such that function $f(x)=\frac{x+a}{x+b}$ is its own inverse. $\left[a\neq -1, b=-1\right]$

1.4 - Quadratic Functions

Topic: Quadratic Functions, Polynomial End Behavior, and Multiplicity

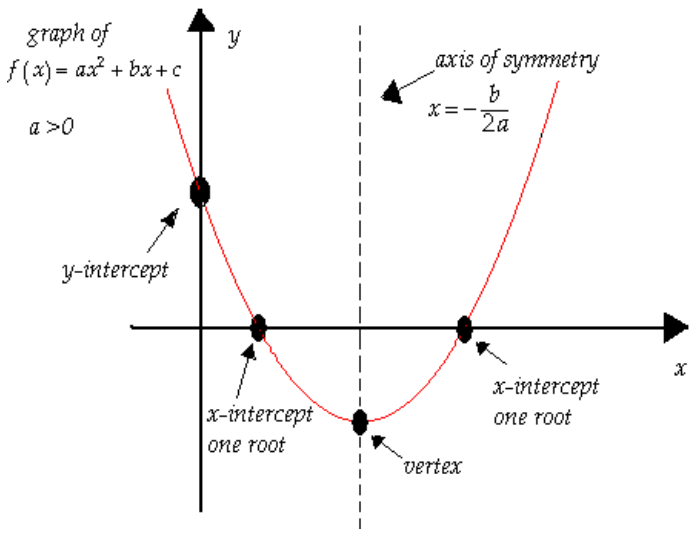
Review

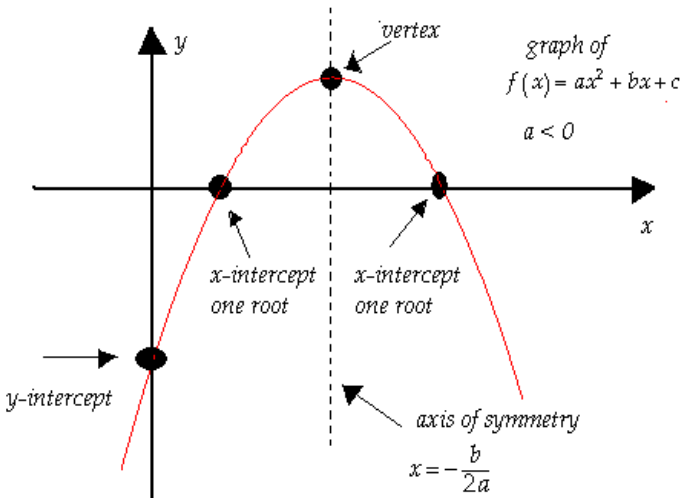
1. The area under $f(x)=bx+ax^2$ on $[0,3]$ is 10. The average rate of change of f on $[0,3]$ is 5. Find a, b and the average value m of f on $[0,3]$. $\left[a=\frac{31}{9}, b=-\frac{16}{3}, m=\frac{10}{3}\right]$
2. Let $f(x)=ax^3+bx$ on $[-1,2]$. The average value of f is 3. The average rate of change is 4, find a,b and the area under the curve on [-1, 2]. $[a=-4, b=16, area=9]$
3. Given $f(x)=\sqrt{3x+5}$ and $g(x)=ax+b$. Find a and b if given that $(f\circ g)(2)=5$ and $g(1)=4$ $[a=2, b=2]$
4. Given $f(g(x))=2x^2-3x+7$ and $g(x)=x-2$, find $f(x+1)$ $[f(x+1)=2x^2+5x+9]$

Key Properties

A function f is a quadratic functions if $f(x)=ax^2+bx+c$ where a,b and c are real numbers.

In the two below diagrams we can see the graph of a quadratic function $f(x)=ax^2+bx+c$





The curve of a quadratic equation is called *parabola*. There are some major terms, shown on the graphs, that we have to consider when graphing a quadratic equation or function.

VALUES of *a*:
if *a* is positive $a > 0$ then the parabola is open upward as in the first diagram.

if *a* is negative $a < 0$ then the parabola is open downward as in the second diagram

***y*-intercept:**
where the curve crosses the *y*-axis. When the curve intersects the *y*-axis the value of *x* is zero so we have $f(0) = c$. If $c = 0$ then the curve passes through the origin.

***x*-intercepts *or ROOTS*:**
where the curve crosses the *x*-axis. When the curve intersects the *x*-axis the value of *f(x)* or *y* is zero we have $ax^2 + bx + c = 0$. If $b = 0$ and $c = 0$ the curve passes through the origin. If the graph lies above the *x*-axis there are no roots. The roots are r_1 and r_2.

AXIS of SYMMETRY:
the graph of a quadratic is symmetric about a vertical line. It means if the above graphs were folded in half along the *y*-axis, the two halves of the parabola would match exactly. The axis of symmetry has equation $x = -\frac{b}{2a}$. If $b = 0$ then the *y*-axis is the axis of symmetry. The axis of symmetry is exactly in between the two roots: $x = \frac{r_1 + r_2}{2} = -\frac{b}{2a}$

VERTEX:
if the parabola is open upward the vertex is the lowest point on the curve.

if the parabola is open downward the vertex is the highest point on the curve.

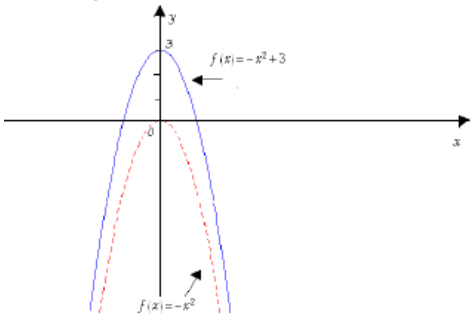
The vertex is always on the axis of symmetry. Its coordinates are always $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

Example 1:

Graph the quadratic function $f(x) = -x^2 + 3$

Solution:

We can see that this is the equation $f(x) = -x^2$ shifted by 3 units upward. Applying the shifting property we have the diagram shown below



It is important to draw accurate graphs by labeling clearly x and y-interecpts, the axis of symmetry and the vertex.

Example 2:

Given $y = 3x^2 - 6x + 4$ find:

(i) y intercept
(ii) x intercepts
(iii) axis of symmetry
(iv) vertex
(v) graph the equation

Solution:

(i) y-intercept when $x = 0$ therefore $y = 4$

(ii) x-intercepts when $y = 0$ so $y = 3x^2 - 6x + 4$ $\quad roots = \dfrac{-(-6) \pm \sqrt{36 - 48}}{6} = \dfrac{6 \pm \sqrt{-12}}{6}$

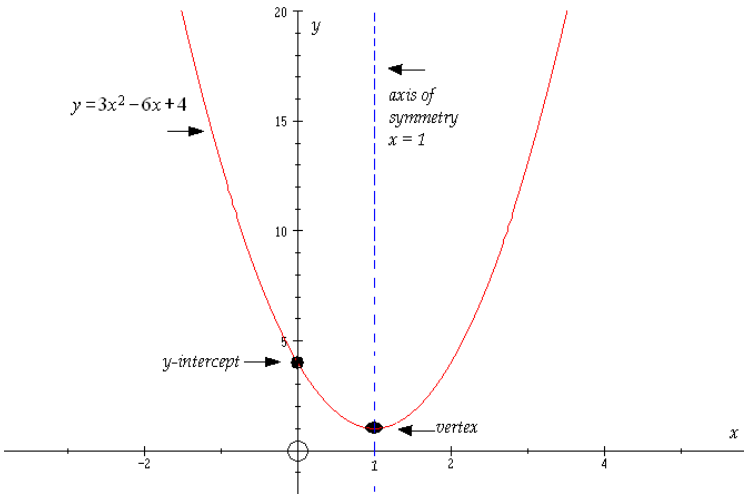
since the value under the square root (discriminant) is negative there are no real values. So there are no x-intercepts.

(iii) axis of symmetry is $\quad x = -\dfrac{b}{2a} = -\dfrac{-6}{2(3)} = \dfrac{6}{6} = 1$

(iv) vertex: we substitute the value of x in the axis of symmetry in y to have

$y = 3(1)^2 - 6(1) + 4 = 3 - 6 + 4 = 1$ so the coordinates of the vertex are (1, 1). Lowest point.

(v) graph: since $a = 3 > 0$ the parabola is open upward



Example 3:

Given $y = -2x^2 - 8x + 3$ find:

(i) y intercept
(ii) x intercepts
(iii) axis of symmetry
(iv) vertex
(v) graph the equation

Solution:

(i) y-intercept when $x = 0$ therefore $y = 3$

(ii) x-intercepts when $y = 0$ so $-2x^2 - 8x + 3 = 0$ $\quad roots = \dfrac{8 \pm \sqrt{64+24}}{-4} = \dfrac{8 \pm \sqrt{88}}{-4}$

$r_1 = -\dfrac{4+\sqrt{22}}{2} \quad r_2 = \dfrac{\sqrt{22}-4}{2}$

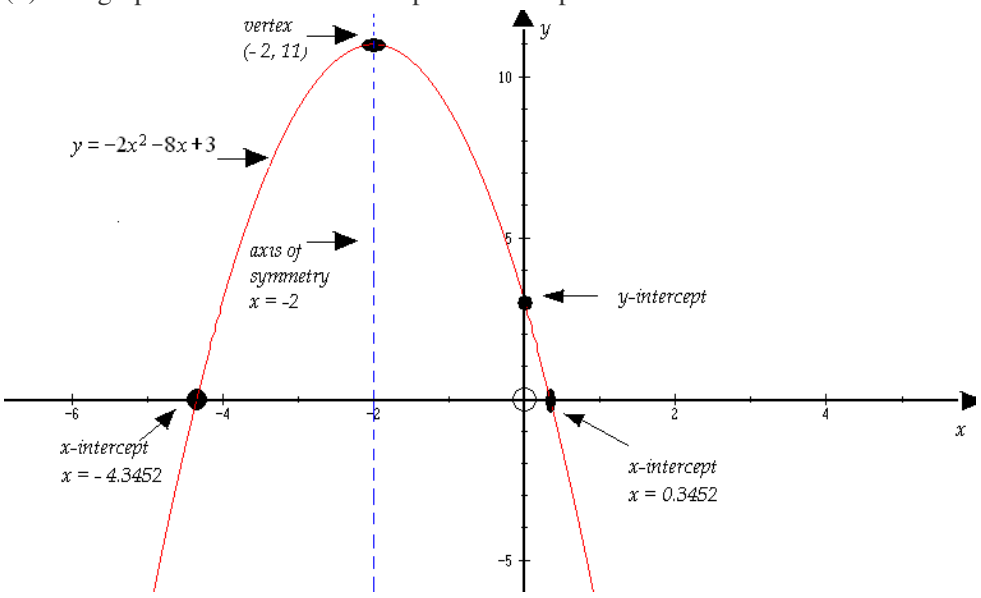
(iii) axis of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{-8}{2(-2)} = -\dfrac{-8}{-4} = -2$

(iv) vertex: we substitute the value of x in the axis of symmetry in y to have

$y = -2(-2)^2 - 8(-2) + 3 = -2(4) + 16 + 3 = 11$

so the coordinates of the vertex are *(-2, 11)*. Highest point.

(v) graph: since $a = -2 < 0$ the parabola is open downward



How to Write a Parabola

A parabola can be written in three ways:

(a) **standard form:** $y = ax^2 + bx + c$

(b) **factor form:** $y = a(x - r_1)(x - r_2)$ where r_1 *and* r_2 are the two roots

(c) **vertex form**: $y = a(x - p)^2 + q$ where p is the axis of symmetry and q is the y-coordinate of the vertex.

Example 4:

Write in vertex form $y = 2x^2 - 12x + 1$

Solution:

We have $a = 2$, $b = -12$ and $c = 1$.

Find the axis of symmetry: $x=-\frac{b}{2a}=-\frac{-12}{2(2)}=3$

Find the vertex: $y(3)=2(3)^2-12(3)+1=18-36+1=-17$

Then $y=2(x-3)^2-17$

Example 5:

Write in vertex form $y=-(x-2)(x+6)$

Solution:

We have $a = -1$.

Roots are $r_1=2,\ r_2=-6$

Find the axis of symmetry: $x=\frac{r_1+r_2}{2}=\frac{-6+2}{2}=-2$

Find the vertex: $y(-2)=-(-2-2)(-2+6)=-(-4)(4)=16$

Then $y=-(x+2)^2+16$

Example 6:

Write in factor form $y=2\left(x-\frac{5}{4}\right)^2-\frac{49}{8}$

Solution:

Expand: $y=2\left(x-\frac{5}{4}\right)^2-\frac{49}{8}=2\left(x^2-\frac{5}{2}x+\frac{25}{16}\right)-\frac{49}{8}=2x^2-5x+\frac{25}{8}-\frac{49}{8}=2x^2-5x-3$

Factor: $y=2x^2-5x-3=(2x+1)(x-3)=2\left(x+\frac{1}{2}\right)(x-3)$

Example 7:

Write the standard form $y=ax^2+bx+c$ given vertex at $(-2,5)$ and y-intercept at -3.

Solution:

Given axis of symmetry: $x = -2$ and vertex $y = 5$ we have in vertex form: $y=a(x+2)^2+5$

y-intercept means: when $x=0, y=-3$.

Substitute to find a:

$-3=a(0+2)^2+5$

$-3=4a+5$

$-8=4a$

$a=-2$

Then: $y=-2(x+2)^2+5=-2(x^2+4x+4)+5=-2x^2-8x-3$

Example 8:

Write the factor form $y=a(x-r_1)(x-r_2)$ given vertex at $(4,-2)$ and y-intercept at 6.

Solution:

Given axis of symmetry: $x = 4$ and vertex $y = -2$ we have in vertex form: $y=a(x-4)^2-2$

y-intercept means: when $x=0, y=6$.

Substitute to find a:

$6=a(0-4)^2-2$

$6=16a-2$

$8=16a$

$a=\frac{1}{2}$

Then: $y=\frac{1}{2}(x-4)^2-2=\frac{1}{2}(x^2-8x+16)-2=\frac{1}{2}(x^2-8x+16-4)=\frac{1}{2}(x^2-8x+12)=\frac{1}{2}(x-6)(x-2)$

Example 9:

Write the standard form of a parabola given vertex at $(3,-4)$ that passes through the point $(5,8)$.

Solution:

Given axis of symmetry: $x = 3$ and vertex $y = -4$ we have in vertex form: $y=a(x-3)^2-4$

point $(5,8)$ means: when $x=5, y=8$.

Substitute to find a:

$8=a(5-3)^2-4$

$12=4a$

$a=3$

Then: $y=3(x-3)^2-4=3(x^2-6x+9)-4=3x^2-18x+23$

Example 10:

The parabola $y=ax^2+bx+c$ open upward, has vertex at $(2,-1)$, x-intercepts at $x = 1$ and $x = 3$.

(i) Determine the values of a,b,c

(ii) Factorize the quadratic ax^2+bx+c

(iii) Find the range (interval) for which ax^2+bx+c is greater than 0

(iv) Find the minimum value of ax^2+bx+c.

Solution:

(i) vertex form $y=a(x-2)^2-1$

Using x-intercepts $(1,0)$: $0=a(1-2)^2-1 \rightarrow 0=a-1 \rightarrow a=1$

So $y=1(x-2)^2-1=x^2-4x+3$.

Therefore $a=1, b=-4, c=3$

(ii) Factorize: $x^2-4x+3=(x-1)(x-3)$

(iii) greater than 0 when $x<1$ or $x>3$

(iv) Find the minimum value at the vertex: minimum = -1

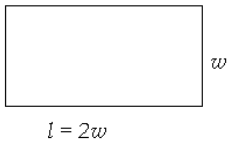
Applications of Quadratic Functions

When expressing a relationship for solving certain problems, we can obtain an equation having more than one solution, therefore we need to distinguish between *possible solutions* and *actual solution.* Each solution must be meaningful to the problem.

Example 11:

The length of a rectangle is twice its width. Express the area of the rectangle as a function of its width. if the area is 338cm^2. find the perimeter

Solution:



draw a diagram and let w = *width* and l = $2w$ = *length*

the area of a rectangle is $\text{Area}=A=width\times length$

since $l = 2w$ we have $A(w)=w\times 2w=2w^2$

we substitute 338 for the area and solve for w: $338=2w^2$ then $169=w^2$

taking the square root we have $w=\pm 13$. Since a distance cannot be negative we have *13cm* as actual solution for w. It follows that $l = 2w$=*26cm.*

Then the perimeter p is $p=2(l+w)=2(26+13)=78cm$

Example 12:

The lengths of the sides of three squares are x, $x + 1$, and $x + 2$. If their total area is 365cm^2 , find their dimensions.

Solution:

The area of the first square is: x^2

The area of the second square is: $(x+1)^2=x^2+2x+1$

The area of the third square is: $(x+2)^2=x^2+4x+4$

The sum of their areas is equal to 365 so we have: $x^2+(x^2+2x+1)+(x^2+4x+4)=3x^2+6x+5=365$

Now we can solve for x:

$3x^2+6x+5=365 \rightarrow 3x^2+6x-360=0 \rightarrow x^2+2x-120=0\rightarrow$

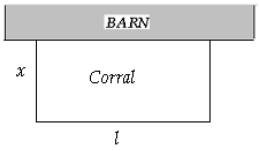
$\rightarrow (x-10)(x+12)=0$

$\rightarrow x=10$ or $x=-12$

We cannot have a negative dimension so:
the dimension of the first square is $x = 10$
the dimension of the second square is $x + 1 = 11$
the dimension of the third square is $x + 2 = 12$

Example 13:

A farmer has 100m of fencing with which to make a rectangular corral. A side of a barn will be used as one side of the corral, as shown in the diagram.



(i) if the width of the corral is x, express the length l in terms of x
(ii) if the width of the corral is x, express the area A in terms of x
(iii) find the dimension of x that gives the corral the greatest possible area
Solution:
(i) there are three sides on the corral x, l and x. Since there are 100m of fencing available we have
$x + l + x = 100$
solving for l: $l = 100 - 2x$

(ii) the area A of the corral $Area = A = width \times lenght$
since $l = 100 - 2x$ we have $A(x) = l \times x = x(100 - 2x) = 100x - 2x^2$

(iii) The function $A(x)$ is a quadratic with $a = -2$ therefore the parabola is open downward
from $100 - 2x$ it follows that x-intercepts are 0 and 50
since $c = 0$ then y-interecpt is 0
axis of symmetry is $x = -\frac{b}{2a} = -\frac{100}{2(-2)} = 25$
Since we have a quadratic function the maximum is the vertex which occurs when x is on the axis of symmetry. Therefore to find the maximum area we substitute $x = 25$ on the function $A(x)$ to have
$A(25) = 100(25) - 2(25)^2 = 1250m^2$

Example 14:

A small shops sells handmade candles. The price (p in dollars) is related to the quantity q sold by the linear demand equation $p = 20 - \frac{1}{2}q$. Write the revenue function $R(q)$ in the form $R(q) = p \times q$.Find:
(i) the value of q that gives the maximum revenue
(ii) the maximum revenue
Solution:
(i) $R(q) = p \times q = \left(20 - \frac{1}{2}q\right)q = 20q - \frac{1}{2}q^2$
It is a quadratic and the maximum occurs on the axis of symmetry:

$$q = -\frac{b}{2a} = -\frac{20}{2\left(\frac{1}{2}\right)} = 20$$

(ii) Maximum revenue: $R(20) = 20(20) - \frac{1}{2}(20)^2 = 400 - 200 = 200$

Example 15:

When a company sells x units of product, its profit is approximated by the function $P(x) = -2x^2 + 40x + 280$. Find

(i) the number of units that should be sold so that maximum profit is received

(ii) the maximum profit

Solution:

(i) The function $P(x)$ is a quadratic, therefore the maximum occurs at the value of x at the axis of symmetry . We have $a = -2$ and $b = 40$, axis of symmetry is $x = -\frac{b}{2a} = -\frac{40}{2(-2)} = 10$.

Therefore the 10 units should be sold to have the maximum profit

(ii) to find the maximum profit we substitute $x = 10$ on the function $P(x)$ to have

$P(10) = -2(10)^2 + 40(10) + 280 = 480$

Example 16:

A theater sells tickets for a concert. When the ticket price is \$40, the theater sells 800 tickets. For every \$2 increase in price, the number of tickets sold decreases by 20. The cost per attendee (staff, cleaning, etc.) is \$10, and fixed costs are \$2000.

(a) Let x be the number of \$2 increases.

Write an expression for the ticket price and the number of tickets sold.

(b) Write the revenue function $R(x)$ (total ticket sales income).

(c) Write the profit function $P(x)$.

(d) Find the price per ticket that maximizes profit, and the maximum profit.

Solution:

(a) Price: $p = 40 + 2x$

Tickets sold: $n = 800 - 20x$ (valid for $0 \le x \le 40$ since $n \ge 0$)

(b) Revenue = price × tickets:

$R(x) = (40 + 2x)(800 - 20x) = -40x^2 + 800x + 32000$

(c) Total cost = variable cost + fixed cost:

Variable = \$10 per attendee × $n = 10(800 - 20x) = 8000 - 20x$

Fixed = 2000

So $C(x) = 8000 - 20x + 2000 = 10000 - 200x$

Profit: $P(x) = R(x) - C(x)$

$P(x) = R(x) = \left(-40x^2 + 800x + 32000\right) - (10000 - 200x) = -40x^2 + 1000x + 22000$

(d) Maximize: vertex at $x = -\frac{b}{2a} = -\frac{1000}{2(40)} = 12.5$

Number of \$2 increases = $= 2(12.5) = 25\,\$$

Price = $40 + 2(12.5) = 65\$$

Tickets sold = $800 - 20(12.5) = 550$

Max profit = $P(12.5) = -40(12.5)^2 + 1000(12.5) + 22000 = 28250\$$

Miscellaneous Problems

Remember the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac > 0 \rightarrow$ two real roots

$b^2 - 4ac = 0 \rightarrow$ one real root or two equal roots

$b^2 - 4ac < 0 \rightarrow$ no real roots

Example 17:

The quadratic equation $x^2 - 8x + k = 0$ has two equal roots. Find the value of k.

Solution

Having two equal roots implies that $b^2 - 4ac = 0 \rightarrow (8)^2 - 4(1)(k) = 0 \rightarrow -4k = -64 \rightarrow k = 16$

Example 18:

The quadratic equation $2x^2 + kx + k = 0$ has no real roots. Find the range of possible values of k.

Solution

Having no real roots implies that $b^2 - 4ac < 0 \rightarrow (k)^2 - 4(2)(k) < 0 \rightarrow k^2 - 8k < 0$

$k^2 - 8k$ is a parabola open upward with roots 0 and 8 and is negative when is between those two values.

Therefore $0 < k < 8$

Example 19:

The quadratic equation $x^2+(k-1)x+(k-2)=0$ has two real roots where one root is twice the other one.

Find all the possible values of k.

Solution

Let r and $2r$ be the two roots then:

$x^2+(k-1)x+(k-2)=0$

$(x-r)(x-2r)=x^2-3r+2r^2=0$

Equalize coefficients: $\begin{cases}(k-1)=-3r\\(k-2)=2r^2\end{cases}$

Solving the system: $\rightarrow\begin{cases}k=1-3r\\(k-2)=2r^2\end{cases}\rightarrow\begin{cases}k=1-3r\\(1-3r-2)=2r^2\end{cases}\rightarrow\begin{cases}k=1-3r\\2r^2+3r+1=0\end{cases}\rightarrow$

$\rightarrow\begin{cases}k=1-3r\\(2r+1)(r+1)=0\end{cases}\rightarrow\begin{cases}k=1-3r\\r=-\frac{1}{2},r=-1\end{cases}$

To find k:

when $r=-\frac{1}{2}$ we have $k=1-3\left(-\frac{1}{2}\right)=\frac{5}{2}$

when $r=-1$ we have $k=1-3(-1)=4$

Example 20:

A parabola opens downward and has a vertex at (2, 9), with x-intercepts at $x=-1$ and $x=5$.

(i) Determine the values of a, b, and c for the standard form $y=ax^2+bx+c$.

(ii) Factorize the quadratic expression.

(iii) Find the interval for which the function is greater than 0.

(iv) Find the y-intercept of the parabola.

Solution

(i) Vertex form $y=a(x-p)^2+q=a(x-2)^2+9$; Using x-intercept $(5,0)$

$0=a(5-2)^2+9\rightarrow 0=9a+9\rightarrow a=-1\quad\rightarrow\quad y=-(x-2)^2+9=-x^2+4x+5$

Therefore: $a=-1,b=4,c=5$

(ii) Factorize: $y=-x^2+4x+1=-(x-5)(x+1)$

(iii) Since the parabola is open downward, it is open between roots: $-1<x<5$

(iv) y-intercept is $c=5$

Example 21:

A landscaper has 120 meters of decorative fencing to enclose a rectangular garden. One side of the garden will be bordered by a straight stone wall, so no fencing is required for that side.

(i) If the width of the garden perpendicular to the wall is x, express the length l (the side parallel to the wall) in terms of x.

(ii) Express the area A of the garden as a function of x.

(iii) Find the dimensions that provide the garden with the greatest possible area and the maximum area.

Solution

(i) Fencing = $x + l + x = 120 \Rightarrow 2x + l = 120 \Rightarrow l = 120 - 2x$

(ii) $A(x) = \text{lenght} \times \text{width} = x\left(120 - 2x^2\right) = 120x - 2x^2$

(iii) $A(x)$ is a parabola opening downward. Max occurs at axis of symmetry $x = -\frac{b}{2a} = -\frac{120}{2(-2)} = 30$

Width $x = 30m$; *Length* $l = 120 - 2(30) = 60$m. ; Max Area = $30 \times 60 = 1800m^2$

Example 22:

A bakery sells gourmet cupcakes. When the price is \$5 per cupcake, they sell 200 cupcakes per week. For every \$0.50 increase in price, the number of cupcakes sold decreases by 10. The cost to produce each cupcake (ingredients, labor, etc.) is \$1.50, and the bakery has fixed weekly costs of \$300.

Let x be the number of \$0.50 price increases.

(a) Write expressions for the price per cupcake and the number of cupcakes sold.

(b) Write the revenue function $R(x)$.

(c) Write the profit function $P(x)$.

(d) Find the price per cupcake that maximizes profit, and calculate the maximum profit.

Solution

(a) Price = $5 + 0.5x$; Quantity = $200 - 10x$

(b) $R(x) = (5 + 0.5x)(200 - 10x) = -5x^2 + 50x + 1000$

(c) Cost = $1.5(200 - 10x) + 300 = 300 - 15x + 300 = 600 - 15x$

Profit = Revenue - Cost = $\left(-5x^2 + 50x + 1000\right) - \left(600 - 15x\right) = -5x^2 + 65x + 400$

(d) Axis of symmetry: $x = -\frac{b}{2a} = -\frac{65}{2(-5)} = \frac{13}{2} = 6.5$

Price = $5 + 0.5(6.5) = 5 + 3.25 = \$8.25$

Max Profit: $P(x) = -5(6.5)^2 + 65(6.5) + 400 = 611.25\$$

Practice Problems

Identify the vertex, axis of symmetry, min/max value, y-intercept, and x-intercepts of each.

1. $y=-x^2+x+56$ $\left[-\frac{1}{2},\frac{225}{4}\right]$; axis sym $x=-\frac{1}{2}$; max= $\frac{225}{4}$;y-int=56; x-int:8 and22
2. $y=2x^2-12x+23$ $[3,5]$; axis sym $x=3$; min=5 ;y-int=23; x-int:none
3. $y=1-x^2$ $[0,1]$; axis sym $x=0$; max=1 ;y-int=1; x-int: ± 1
4. $y=-\frac{4}{5}(x+6)(x+5)$ $\left[-\frac{11}{2},\frac{1}{5}\right]$; *axis sym* $x=-\frac{11}{2}$; *max=* $\frac{1}{5}$ *;y-int=-24; x-int:-6 and -5*

Write in vertex form

5. $y=-2x^2+8x-10$ $\left[y=-2(x-2)^2-2\right]$
6. $y=(x-3)(x-6)$ $\left[y=\left(x-\frac{9}{2}\right)^2-\frac{9}{4}\right]$
7. $y=\frac{1}{4}(x-3)(x-2)$ $\left[y=\frac{1}{4}\left(x-\frac{5}{2}\right)^2-\frac{1}{16}\right]$
8. $y=3x(x+3)$ $\left[y=3\left(x+\frac{3}{2}\right)^2-\frac{27}{4}\right]$

Use the information provided to write the standard form equation of each parabola.

9. Vertex (-3, 3), y-int: -6 $\left[y=-x^2-6x-6\right]$
10. Vertex (-6, 9), y-int: -27 $\left[y=-x^2-12x-27\right]$
11. Vertex (5, 3), y-int: 178 $\left[y=7x^2-70x+178\right]$
12. Vertex (-4, -8), y-int: -12 $\left[y=-\frac{1}{4}x^2-2x-12\right]$

Use the information provided to write the factor form equation of each parabola.

13. Vertex (6, 4), y-int: -32 $\left[y=-(x-8)(x-4)\right]$
14. Vertex $\left(\frac{7}{2},\frac{9}{8}\right)$, y-int: -5 $\left[y=-\frac{1}{2}(x-5)(x-2)\right]$
15. Vertex $\left(-\frac{3}{2},\frac{1}{2}\right)$, y-int: -4 $\left[y=-2(x+2)(x+1)\right]$
16. Vertex $\left(-\frac{13}{2},\frac{1}{16}\right)$, y-int: $-\frac{21}{2}$ $\left[y=-\frac{1}{4}(x+7)(x+6)\right]$

Use the information provided to write the standard form equation of each parabola.

17. Open up or down, Vertex $(-1,-2)$, passes through $(0,-16)$ $\quad \left[y=-14x^2-28x-16\right]$
18. Open up or down, Vertex $(6,5)$, passes through $(3,-157)$ $\quad \left[y=-18x^2+216x-643\right]$
19. Open up or down, Vertex $(9,-3)$, passes through $(10,0)$ $\quad \left[y=3x^2-54x+240\right]$
20. Open up or down, Vertex $(-1,0)$, passes through $(-4,9)$ $\quad \left[y=-x^2-2x-1\right]$

21. given the graph find the quadratic function $f(x)=ax^2+bx+c$

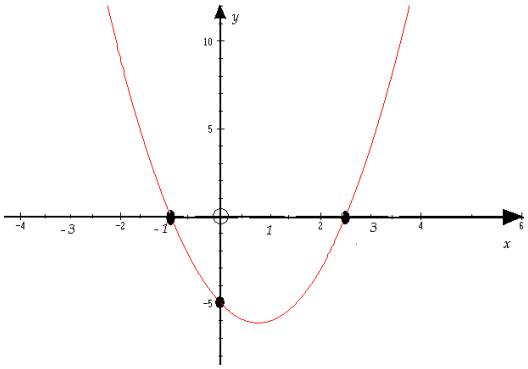
The graph of $y=ax^2+bx+c$ is shown:

(i) determine the value of a , b and c

(ii) factorize the quadratic ax^2+bx+c

(i) find the range (interval) for which ax^2+bx+c is less than 0

(iv) find the minimum value of ax^2+bx+c



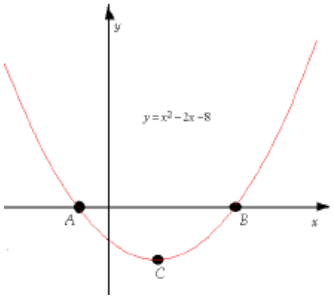
(i) $a = 2,\ b = -3,\ c = -5$

(ii) $y=2x^2-3x-5=(2x-5)(x+$

(iii) $-1<x<\dfrac{5}{2}$

(iv) $\min=-\dfrac{49}{8}$

22. The diagram shows the graph of $y=x^2-2x-8$. The graph crosses the x-axis at the points A and B, and has a vertex at C.



(a) Factorize x^2-2x-8

(b) Write down the coordinates of each of these points

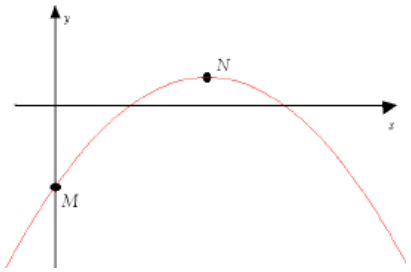
(i) A

Ans:

(a) $(x+2)(x-4)$

(b) A=(-2,0), B=(4,0)

23. The diagram below shows part of the graph of $y = ax^2 + 4x - 3$. The line $x = 2$ is the axis of symmetry. *M* and *N* are points on the curve. *M* is the y-intercept and N is the vertex.



(a) find the value of *a*

(b) find the coordinates of

(i) *M*

(ii) *N*

Ans:

(a) *a= - 1*

(b) (i) *M* = *y*-intercept = *(0 , - 3)*
(ii) *N* = (2, 3)

24. The maximum value of the function $f(x) = ax^2 + bx + c$ is 10. Given that $f(3) = f(-1) = 2$, find $f(2)$. [8]

25. Write the standard equation of the parabola through points (-1,3) and (2,0) having the axis of symmetry the vertical line *x=1*. [$f(x) = x^2 - 2x$]

26. Write the standard equation of the parabola through the points (0,1), (-2,7) and (2,3).
[$x^2 - x + 1$]

27. Show that the slope of a tangent line of the parabola $y = ax^2 + bx + c$ at $x = k$ is $m = 2xk + b$.

28. Write the standard equation of the parabola having vertex at (2,-4) and tangent to the line *y=2x-9*.
[$x^2 - 4x$]

29. The maximum value of the function $f(x) = ax^2 + bx + c$ is 16. Given that $f(0) = f(4) = 12$, find $f(5)$ $[f(5) = 7]$

30. When a company sells *x* units of product, its profit is approximated by the function $P(x) = -2x^2 + 40x + 280$. Find

(i) the number of units that should be sold so that maximum profit is received [10 units]

(ii) the maximum profit [480]

31. Suppose that the price and demand for an item are related by $p(x) = 500 - x^2$ (demand function) Where *p* is the price and *x* is the number of items demanded. The price and supply are related by $p(x) = x^2 - 6x + 140$ (supply function)

where *x* is the supply of the item. Find:

(i) the equilibrium demand and supply [15]

(ii) and the equilibrium price [275]

32. A local music venue hosts a concert. When the ticket price is \$60, they sell 400 tickets. For every \$5 increase in price, the number of tickets sold decreases by 20. The operating cost per attendee is \$15, and the venue has fixed costs of \$2,500. Let x be the number of 5 price increases.
(a) Write expressions for the ticket price and the number of tickets sold. [*60 + 50x; 400-20x*]
(b) Write the revenue function *R(x)*. $\left[R(x) = -100x^2 + 800x + 24000\right]$
(c) Write the profit function *P(x)*. $\left[P(x) = -100x^2 + 1100x + 15500\right]$
(d) Find the ticket price that maximizes profit and calculate the maximum profit.
$\left[price = 87.5; MaxP(5.5) = 18525\$\right]$

33. The lengths of the sides of three squares are $x, x+3$ and $x+5$. If their total area is $683cm^2$, find the dimensions of each square. $[13,15,17]$

34. A rectangular field are 792m^2 is enclosed by 116m of fencing. Find the dimensions of the field. [22 and 36]

35. Given $y = ax^2 + bx + c$, show that the sum of their roots r_1 and r_2 is $r_1 + r_2 = -\dfrac{b}{a}$

36. The quadratic equation $(k-1)x^2 + 2kx + (k+3) = 0$ has real roots *a* and *b* such that $a^2 + b^2 = 6$
Find the possible value(s) of k. $[k = 0; k = 2]$

37. The quadratic equation $(2k+1)x^2 - 6kx + (2k+1) = 0$ has two real roots that are reciprocals of each other and the sum of the roots is 2. Find the value(s) of k. $[k = 1]$

38. The quadratic equation $kx^2 - 5x + (2k-3) = 0$ has roots that are reciprocals of each other. Find the value(s) of k. $[k = 3]$

1.5-Polynomial Functions

Topic: Polynomial Theorems, Long Division vs. Synthetic Division

Review

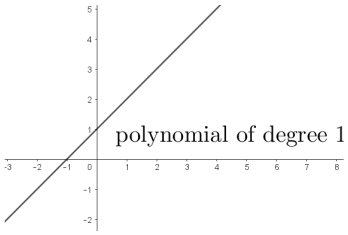
1. The maximum value of the function $f(x)=ax^2+bx+c$ is 12. Given that $f(2)=f(6)=4$, find $f(4)$. $[f(4)=12]$
2. The quadratic equation $(k+2)x^2-4x+(k-1)=0$ has real roots a and b such that $a^2+b^2=10$ Find the possible value(s) of k. $[k=0, k=2]$
3. Prove that for any quadratic equation $ax^2+bx+c=0$ the sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$
4. Write the standard equation of the parabola having vertex at (-1,3) and tangent to the line $y= 1- x$. $[y=x^2+2x+4]$

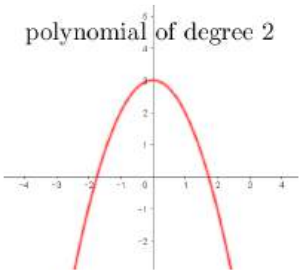
Definition Polynomial Functions

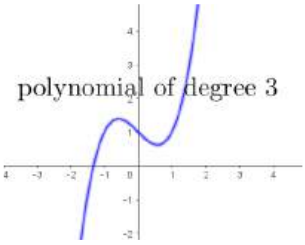
A polynomial function of degree n is a function of the form $p(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1x+a_0$

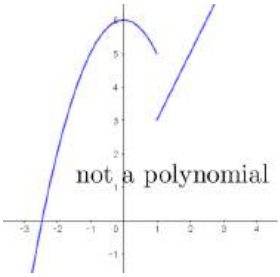
a_0 is the constant term, a_n is the leading coefficient and the term a_nx^n is the leading term.

Polynomials are continuous everywhere.









Remainder Theorem

If a polynomial $p(x)$ is divided by $(x-c)$, where c is a real number, then the remainder is $R(c)$.
In other words, $p(x)=(x-c)\times q(x)+R(c)$ where $q(x)$ is the quotient polynomial. If the remainder is zero then $(x-c)$ is a factor of $p(x)$.

When you divide a polynomial by a linear divisor of the form $(x-c)$, the division algorithm tells us that the remainder is a constant (because the divisor is degree 1). The Remainder Theorem states that you simply evaluate the polynomial at $x=c$ The number you get is the remainder.

Example 1:

Find the remainder when $p(x)=x^2-3x+5$ is divided by $x-2$

Solution:

Using the remainder theorem:

$p(2)=(2)^2-3(2)+5=4-6+5=3$.

Therefore the remainder is 3.

Example 2:

Find the remainder when $p(x)=2x^3+4x^2-x+7$ is divided by $x+3$

Solution:

Note that $x+3=x-(-3)$ so $c=-3$

Using the remainder theorem:

$p(-3)=2(-3)^3+4(-3)^2-(-3)+7=-54+36+3+7=-8$.

Therefore the remainder is - 8.

Example 3:

The remainder when $p(x)=x^3+ax^2+bx+6$ is divided by $x-1$ is 10, and when divided by $x+2$ the remainder is 0. Find a and b.

Solution:

The remainder theorem for $x-1$:

$p(1)=(1)^3+a(1)^2+b(1)+6=1+a+b+6=a+b+7=10 \rightarrow a+b=3$.

The remainder theorem for $x+2$:

$p(-2)=(-2)^3+a(-2)^2+b(-2)+6=-8+4a-2b+6=4a-2b-2=0 \rightarrow 2a-b=1$.

Solve the system: $\begin{cases}2a-b=1\\a+b=3\end{cases}$ Add equations $\rightarrow 3a=4 \rightarrow a=\frac{4}{3}$ then $b=3-a=3-\frac{4}{3}=\frac{5}{3}$

Long Division vs Synthetic Division

Polynomial long division

It is a systematic algorithm used to divide a polynomial $p(x)$ by a divisor $x-c$ to obtain a quotient and a remainder, similar to numerical long division. To perform the division:
1. first write both polynomials in descending order of their degrees, inserting 0 as a coefficient for any missing powers.
2. Next, divide the first term of the dividend by the first term of the divisor to get the first term of the quotient.
3. Multiply the entire divisor by this term, subtract the result from the dividend, and "bring down" the next term.
4. Repeat this process until the degree of the remaining polynomial is less than the degree of the divisor.

Example 4:

Divide x^2+5x+6 by $x+2$.

Solution:

Divide: take the first term of the dividend $\left(x^2\right)$ and divide by the first term of the divisor (x)

$x^2 \div x = x$. This is the first term of the quotient.

Multiply and Subtract: multiply x by $(x+2)$ to have x^2+2x. Subtract it from the original polynomial:

$\left(x^2+5x+6\right)-\left(x^2+2x\right)=3x+6$. This is a new dividend.

Repeat:

- Divide: $(3x+6)\div x=3$. This is the second term of the quotient.
- Multiply and Subtract: multiply 3 by $(x+2)$ to have $3x+6$. Subtract it from the second dividend polynomial: $(3x+6)-(3x+6)=0$

Since 0 cannot be divided by $x+2$, it is the remainder. So the quotient is $x+3$ and the remainder is 0.

We can write $\left(x^2+5x+6\right)\div(x+2)=x+3$.

Example 5:

Divide $2x^3-3x^2+4x+5$ by $x-1$.

Solution:

Divide: take the first term of the dividend $\left(2x^3\right)$ and divide by the first term of the divisor (x)

$2x^3 \div x = 2x^2$. This is the first term of the quotient.

Multiply and Subtract: multiply $2x^2$ by $(x-1)$ to have $2x^3-2x^2$. Subtract it from the original polynomial: $\left(2x^3-3x^2+4x+5\right)-\left(2x^3-2x^2\right)=-x^2+4x+5$. This is the new dividend.

Repeat:

- Divide: $\left(-x^2+4x+5\right)\div x=-x$. This is the second term of the quotient.
- Multiply and Subtract: multiply $-x$ by $(x-1)$ to have $-x^2+x$. Subtract it from the second dividend polynomial: $\left(-x^2+4x+5\right)-\left(-x^2+x\right)=3x+5$. This is the new dividend

Repeat:

- Divide: $(3x+5)\div x=3$. This is the third term of the quotient.
- Multiply and Subtract: multiply 3 by $(x-1)$ to have $3x-3$. Subtract it from the new dividend polynomial: $(3x+5)-(3x-3)=8$. This is the new dividend

Since 8 cannot be divided by $x-1$, it is the remainder.

So the quotient is $2x^2-x+3$ and the remainder is 8.

We can write $2x^3-3x^2+4x+5=2x^2-x+3+\dfrac{8}{x-1}$.

Synthetic Division

It is a shortcut method for dividing a polynomial by a linear divisor of the form $x-c$. It is faster and more compact than polynomial long division, using only the coefficients of the polynomial and the constant c. The process yields the coefficients of the quotient polynomial and the remainder.

Divide $p(x)$ by $x-c$

1. Write coefficients of $P(x)$ in descending order of degree. Include zeros for any missing terms.
2. Place c to the left (since divisor is $x - c$).
3. Bring down the leading coefficient to the bottom row.
4. Multiply that number by c, write the result under the next coefficient, and add.
5. Repeat multiply/add until all coefficients are used.
6. The last number in the bottom row is the remainder. The preceding numbers are the coefficients of the quotient (one degree less than the original polynomial)

Example 6:

Divide x^3+3x^2-9x-8 by $x+1$ using synthetic division.

Solution:

Coefficients: 1, 3, -9, -8; c = *-1*

-1	1	3	-9	-8
		-1	-2	11
	1	2	-11	3

Bring down 1; $-1\times1=-1$ add to 3 $\rightarrow 2$; $-1\times2=-2$ add to -9 $\rightarrow -11$; $-1\times(-11)=11$ add to -8 $\rightarrow 3$

So we have: quotient $x^2+2x-11$, remainder 3

Example 7:

Divide $2x^3+5x^2-11$ by $x+3$ using synthetic division.

Solution:

Coefficients: 2, 5, 0, -11; c = -3

-3	2	5	0	-11
		-6	3	-9
	2	-1	3	-20

Bring down 2;

$-3\times 2=-6$ add to 5 $\rightarrow -1$;

$-3\times(-1)=3$ add to 0 $\rightarrow 3$

$-3\times 3=-9$ add to -11 $\rightarrow -20$

So we have: quotient $2x^2-x+3$, remainder -20

Example 8:

When $p(x)=x^3+kx^2-4x+2$ is divided by $x-2$ the remainder is 10. Find k.

Solution:

Coefficients: 1, k, -4, 2; c = 2

2	1	k	-4	2
		2	$2k$+4	$4k$
	1	k+2	$2k$	$4k$+2

The remainder is $4k$ +2 and the given remainder is 10 then it follows: $4k+2=10\rightarrow k=2$

Factor Theorem

Let $p(x)$ be a polynomial and c be a real number. The following statements are logically equivalent, meaning if one is true, all are true:

1. c is a zero of $p(x)$
2. $x=c$ is a solution of the equation
3. $(x-c)$ is a factor of $p(x)$
4. c is an $x-$intercept of the graph of $p(x)$
5. when $p(x)$ is divided by $(x-c)$ the remainder is 0.

Example 9:

Given $p(x)=x^2-5x-14$ is $c=-2$ a zero of $p(x)$?

Solution:

Since $p(-2)=(-2)^2-5(-2)-14=4+10-14=0$, $c=-2$ is a zero of $p(x)$

Also $(x-c)$ is a factor. Indeed $p(x)=x^2-5x-14=(x+2)(x-7)$

Example 10:

Determine *a,b,c* and *d* of $p(x)=ax^3+bx^2+cx+d$ given that -1, 1, and 3 are zeros of $p(x)$

Solution:

Since -1, 1, and 3 are zeros of $p(x)$ it follows the $(x-1),(x+1),(x-3)$ are factors of $p(x)$.

Therefore $p(x)=(x-1)(x+1)(x-3)=\left(x^2-1\right)(x-3)=x^3-3x^2-x+3$.

Equalize the coefficients: $ax^3+bx^2+cx+d=x^3-3x^2-x+3$ and we have

$a=1,b=-3,x=-1,d=3$

Example 11:

Given $(x-3)$ is a factor of $p(x)=x^3-4x^2+x+6$, write the complete factorization.

Solution:

Since $(x-3)$ is factor then $x=3$ is a zero of $p(x)$. Using synthetic division:

3	1	- 4	1	6
		3	-3	-6
	1	-1	-2	0

Then $p(x)=x^3-4x^2+x+6=(x-3)(x-x-2)=(x-3)(x-2)(x+1)$

Example 12:

Prove that $(x-c)$ is a factor of $p(x)=x^n-c^n$ for any positive integer *n*.

Solution:

If $(x-c)$ is a factor of $p(x)=x^n-c^n$ then by the factor theorem $x=c$ is a zero.

Then $p(c)=c^n-c^n=0$

If $p(c)=0$ then by the remainder theorem $(x-c)$ is a factor of $p(x)=x^n-c^n$

Rational Zero Theorem

Let $p(x)$ a polynomial with degree *n* then the *possible rational zero*

$p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ are of the form: $\pm\dfrac{\text{factors } a_0}{\text{factors } a_n}$

where

a_0 is the constant term

a_n is the leading coefficient

The Rational Root Theorem provides a finite list of *candidates* for rational roots of a polynomial with integer coefficients. If a candidate *c* gives $p(c)=0$, then $x-c$ is a factor. This theorem does not guarantee that rational roots exist. It only tells you what they could be if they do exist.

Example 13:

Given $p(x)=2x^3+2x^2-3x-3$, state the possible rational zeros. Then find the zeros.

Solution:

We have constant term -3, its factors are $\pm1,\pm3$

We have leading coefficient 2, its factors are $\pm1,\pm2$

Possible rational zeros: $\pm\dfrac{\text{factors constant term}}{\text{factors leading coefficient}}=\pm1,\pm\dfrac{1}{2},\pm3,\pm\dfrac{3}{2}$

Using grouping: $p(x)=2x^3+2x^2-3x-3=2x^2(x+1)-3(x+1)=(2x^2-3)(x+1)$

Zeros are: $x=\pm\dfrac{3}{2},x=-1$

Example 14:

Given $p(x)=2x^3+x^2-5x+2$, state the possible rational zeros. Then find the zeros.

Solution:

We have constant term 2 , its factors are $\pm1,\pm2$; We have leading coefficient 2, its factors are $\pm1,\pm2$

Possible rational zeros: $\pm\dfrac{\text{factors constant term}}{\text{factors leading coefficient}}=\pm1,\pm\dfrac{1}{2},\pm2$

By inspection: $p(1)=5(1)^3-9(1)^2+3(1)+1=0$

Using synthetic division to lower the degree:

1	2	1	-5	2
		2	3	-2
	2	3	-2	0

Then $p(x)=2x^3+x^2-5x+2=(x-1)(2x^2+3x-2)=(x-1)(2x-1)(x+2)$;

Zeros are: $x=-2,\ x=\dfrac{1}{2},\ x=1$

The Conjugate Root Theorem

The conjugate root theorem states that if an irrational number $a+\sqrt{b}$ is a root of a polynomial $p(x)$ in one variable with real coefficients, then the conjugate $a-\sqrt{b}$ is also a root of that polynomial.

Example 15:

A polynomial function $p(x)$ with rational coefficients has $\sqrt{2}$ and 5 as zeros. Find all additional zeros and determine the least degree of $p(x)$

Solution:

Since $\sqrt{2}$ is a root also $-\sqrt{2}$ is a root by the Conjugate Root Theorem.

We have 3 zeros $\pm\sqrt{2}$ and 5 therefore the least degree of $p(x)$ is 3.

Example 16:

Write a polynomial function of least degree with integer coefficients that has the given zeros:

$\sqrt{3}$ and $-\sqrt{5}$

Solution:

By the Conjugate Root Theorem we have two other roots: $-\sqrt{3}$ and $\sqrt{5}$

We have 4 roots therefore the least degree of $p(x)$ is 4.

Then $p(x)$ has the following factors: $\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)$.

Let's multiply them:

$$p(x)=\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)=\left(x^2-3\right)\left(x^2-5\right)=x^4-8x^2+15$$

Example 17:

Write a polynomial function of least degree with integer coefficients that has the given zeros:

$1-\sqrt{7}$ and $\frac{2}{3}$

Solution:

By the Conjugate Root Theorem we have one more roots: $1+\sqrt{7}$

We have 3 roots therefore the least degree of $p(x)$ is 3.

Then $p(x)$ has the following factors: $\left(x-\left(1-\sqrt{7}\right)\right)\left(x-\left(1+\sqrt{7}\right)\right)\left(x-\frac{2}{3}\right)$.

To simplify the multiplication we can factor the negative sign from the irrational roots and rewrite

$\left(x-\left(1-\sqrt{7}\right)\right)$ as $\left[(x-1)+\sqrt{7}\right]$; $\left(x-\left(1+\sqrt{7}\right)\right)$ as $\left[(x-1)-\sqrt{7}\right]$; and $\left(x-\frac{2}{3}\right)$ as $(3x-2)$.

Let's multiply them:

$$p(x)=\left[(x-1)+\sqrt{7}\right]\left[(x-1)-\sqrt{7}\right](3x-2).$$

$$p(x)=\left[(x-1)^2-\left(\sqrt{7}\right)^2\right](3x-2)$$
$$p(x)=\left[x^2-2x+1-7\right](3x-2)$$
$$p(x)=\left[x^2-2x-6\right](3x-2)$$
$$p(x)=\left[x^2-2x-6\right](3x-2)=3x^3-8x^2-14x+12$$

Fundamental Theorem of Algebra (simplified version)

Any polynomial $p(x)$ of degree *n* has *n* roots.

In other words: every polynomial $p(x)$ will have exactly the number of answers as its highest power.

Example 18:

Consider 3 cases:

Case 1: $x^2-9=0$

The highest power is 2 and we have two roots: $x=3,\ x=-3$

Case 2: $x^2-10x+25=0$

which factors $(x-5)^2=0$. While $x = 5$ is the only numerical solution, satisfying the Fundamental Theorem of Algebra requires introducing the concept of multiplicity.

Multiplicity is simply the number of times a specific answer repeats itself in a polynomial equation.

Because $x = 5$ has a multiplicity of 2, it counts twice and the theorem stays undefeated.

Case 3: $x^2+1=0$

The highest power is 2. If you try to solve you get $x^2=-1$. No regular number multiplied by itself equals a negative. The answers are $x=i,\ \ x=-i$. Including complex numbers the Fundamental Theorem of Algebra is satisfied. Complex roots always occur in conjugate pairs (2,4, 6…)

Example 19:

Determine how many roots $p(x)=x^4-81$ has, then find the real roots.

Solution:

The highest power is 4 $\rightarrow$ 4 roots.

Factorize $p(x)=x^4-81=\left(x^2-9\right)\left(x^2+9\right)=(x-3)(x+3)\left(x^2+9\right)$

Since $\left(x^2+9\right)$ does not have real roots. The real roots are $x=3,\ x=-3$.

Example 20:

Determine how many roots $p(x)=x^3-5x^2+4x-20$ has, then find the real roots.

Solution:

The highest power is 3 $\rightarrow$ 3 roots.

Factorize $p(x)=x^3-5x^2+4x-20=x^2(x-5)+4(x-5)=(x^2+4)(x-5)$

Since (x^2+4) does not have real roots. The only real root is $x=5$.

Example 21:

Determine how many roots $p(x)=x^4-3x^3-3x^2+15x-10$ has. Given that $x=2$ is one root, then factorize completely $p(x)$ and find all the real roots and their multiplicity.

Solution:

The highest power is 4 $\rightarrow$ 4 roots.

Using synthetic division:

2	1	-3	-3	15	-10
		2	-2	-10	10
	1	-1	-5	5	0

Factorize $p(x)=x^4-3x^3-3x^2+15x-10=(x-2)(x^3-x^2-5x+5)$

Grouping: $p(x)=x^4-3x^3-3x^2+15x-10=(x-2)\left[x^2(x-1)-5(x-1)\right]=(x^2-5)(x-1)(x-2)$

Difference of two squares:

$p(x)=x^4-3x^3-3x^2+15x-10=(x-2)\left[x^2(x-1)-5(x-1)\right]=(x^2-5)(x-1)(x-2)=(x-\sqrt{5})(x+\sqrt{5})(x-1)(x-2)$

There are 4 real roots: $x=\sqrt{5},\ x=-\sqrt{5},\ x=1,\ x=2$.

Miscellaneous Problems

Example 22:

A polynomial $p(x)$ leaves a remainder of 3 when divided by $(x-1)$ and a remainder of -8 when divided by $(x+1)$. Find the remainder when is divided by the quadratic divisor (x^2-1).

Solution:

When divided by (x^2-1) the remainder must be in the form $R(x)=ax+b$ because the divisor is of degree 2.

When the root is 1 the remainder is 3 $\rightarrow$ $a(1)+b=3\rightarrow a+b=3$

When the root is – 1 the remainder is – 8 $\rightarrow$ $a(-1)+b=-8\rightarrow -a+b=-8$

Solving the system by adding the two equations:

$$\begin{cases} a+b=3 \\ -a+b=-8 \end{cases} \rightarrow 2b=11\rightarrow b=\frac{11}{2}\rightarrow a+\frac{11}{2}=3\rightarrow a=-\frac{5}{2}$$

Example 23:

When $p(x)=x^4+kx^3-4x^2+mx+2$ is divided by $(x-2)$, the remainder is 10. When the same polynomial is divided by $(x+1)$, it is completely divisible (the remainder is 0). Find the values of the constants k and m.

Solution:

When $x=2$ we have $p(2)=10 \rightarrow (2)^4+k(2)^3-4(2)^2+m(2)+2=10 \rightarrow$

$\rightarrow 16+8k-16+2m+2=10 \rightarrow 4k+m=4$

When $x=-1$ we have $p(-1)=0 \rightarrow (-1)^4+k(-1)^3-4(-1)^2+m(-1)+2=0 \rightarrow$

$\rightarrow 1-k-4-m+2=0 \rightarrow -k-m=1$

Solving the system by adding the two equations:

$$\begin{cases} 4k+m=4 \\ -k-m=1 \end{cases} \rightarrow 3k=5 \rightarrow k=\frac{5}{3} \rightarrow \frac{20}{3}+m=4 \rightarrow m=-\frac{8}{3}$$

Example 24:

Find a polynomial function $p(x)$ of the least degree with integer coefficients that has the zeros $x=2,\ x=1+\sqrt{3}$.

Solution:

By the Conjugate Root Theorem we have one more roots: $1-\sqrt{3}$

We have 3 roots therefore the least degree of $p(x)$ is 3.

Then $p(x)$ has the following factors: $\left(x-\left(1-\sqrt{3}\right)\right)\left(x-\left(1+\sqrt{3}\right)\right)(x-2)$.

To simplify the multiplication we can factor the negative sign from the irrational roots:

$\left[(x-1)+\sqrt{3}\right]\left[(x-1)-\sqrt{3}\right]$

Let's multiply them:

$p(x)=\left[(x-1)+\sqrt{3}\right]\left[(x-1)-\sqrt{3}\right](x-2)$

$p(x)=\left[x^2-2x+1-3\right](x-2)$

$p(x)=\left[x^2-2x-2\right](x-2)$

$p(x)=\left[x^2-2x-2\right](x-2)=x^3-4x^2+2x+4$

Example 25:

Consider the polynomial $p(x)=2x^4-x^3-5x^2+2x+2$.

1. State all possible rational zero candidates.
2. By factoring the polynomial completely, find all real roots.

Solution:

1. We have constant term 2, its factors are $\pm 1, \pm 2$

We have leading coefficient 2, its factors are $\pm 1, \pm 2$

Possible rational zeros: $\pm\dfrac{\text{factors constant term}}{\text{factors leading coefficient}}=\pm1,\pm\dfrac{1}{2},\pm2$

2. By inspection $p(1)=2(1)^4-(1)^3-5(1)^2+2(1)+2=2-1-5+2+2=0$

Using synthetic division

1	2	-1	-5	2	2
		2	1	-4	-2
	2	1	-4	-2	0

$p(x)=2x^4-x^3-5x^2+2x+2=(x-1)(2x^3+x^2-4x-2)$

Grouping: $(2x^3+x^2-4x-2)=x^2(2x+1)-2(2x+1)=(x^2-2)(2x+1)$

$p(x)=(x-1)(x^2-2)(2x+1)=(x-1)(x-\sqrt{2})(x+\sqrt{2})(2x+1)$

The roots are: $x=1,\ x=\sqrt{2},\ x=-\sqrt{2},\ x=-\dfrac{1}{2}$

Example 26:

Determine how many roots $p(x)=x^4-6x^3+5x^2+24x-36$ has. Given that $x=2$ is one root, then factorize completely $p(x)$ and find all the real roots and their multiplicity.

Solution:

The highest power is 4 $\rightarrow$ 4 roots.

Apply synthetic division:

2	1	-6	5	24	-36
		2	-8	-6	36
	1	-4	-3	18	0

$p(x)=x^4-6x^3+5x^2+24x-36=(x-2)(x^3-4x^2-3x+18)$

Apply the rational theorem to $(x^3-4x^2-3x+18)$

We have constant term 18 , its factors are $\pm1,\pm18,\pm2,\pm9,3\pm,\pm6$

We have leading coefficient 1, its factors are ±1

Possible rational zeros: $\pm\dfrac{\text{factors constant term}}{\text{factors leading coefficient}}=\pm1,\pm18,\pm2,\pm9,3\pm,\pm6$

By inspection:

$p(1)=(1)^3-4(1)^2-3(1)+18=12 \rightarrow (x-1)$ is not a factor

$p(-2)=(-2)^3-4(-2)^2-3(-2)+18=-8-16+6+18=0 \rightarrow (x+2)$ is a factor

Apply synthetic division again to $(x^3-4x^2-3x+18)$:

-2	1	-4	-3	18
		-2	12	-18-
	1	-6	9	0

$p(x)=x^4-6x^3+5x^2+24x-36=(x-2)(x+2)(x^2-6x+9)$

But $(x^2-6x+9)=(x-3)^2$ then

$p(x)=x^4-6x^3+5x^2+24x-36=(x-2)(x+2)(x-3)^2$

The roots are: $x=2,\ x=-2,\ (x=3$ with multiplicity $2)$

Practice Problems

1. Find remainder when $x^3 - 4x^2 + 2x - 5$ is divided by $\boldsymbol{x - 3}$. [-8]
2. Find remainder when $2x^4 - 3x^3 + x^2 - 6$ is divided by $x + 2$. [54]
3. If $P(x) = x^5 - kx^3 + 2x - 1$ has remainder 7 when divided by $x - 2$, find k [6]
4. The remainder when $ax^3 + bx^2 - 3x + 4$ is divided by $x - 1$ is 6, and by $x + 1$ is 2. Find a, b. [$a=3,\ b=0$]
5. When $P(x)$ is divided by $x - 3$, remainder 5; by $x + 4$, remainder -2. Find remainder when divided by $(x - 3)(x + 4)$. [$x+2$]
6. Divide $x^3 - 6x^2 + 11x - 6$ by $x - 2$. Write quotient and remainder
   $\left[q(x) = x^2 - 4x + 3; R(x) = 0\right]$
7. Divide $2x^3 - 5x^2 + 3x - 7$ by $x - 3$. Write quotient and remainder.
   $\left[q(x) = 2x^2 + x + 6; R(x) = 11\right]$
8. Find k so that $x^4 - kx^3 + 2x^2 - kx + 1$ is divisible by $x - 2$. $\left[k = \frac{29}{9}\right]$
9. When $x^3 + ax^2 + bx + 6$ is divided by $x - 2$, remainder 10; by $x + 1$, remainder -8. Find a, b. [$a=3,\ b=4$]
10. Show that $x - 2$ is a factor of $x^3 - 5x^2 + 8x - 4$ and find the other factor. $\left[x^2 - 3x + 2\right]$
11. Factor $x^3 - 3x^2 - 13x + 15$ given $x - 5$ is a factor. $\left[(x-5)(x+3)(x-1)\right]$
12. Find k so that $x^3 - kx^2 + 5x - 2$ has $x - 1$ as a factor. $\left[k = 4\right]$
13. Given $x + 3$ is a factor of $x^4 - 2x^3 - 13x^2 + 14x + 24$, find all factors.
    $\left[(x-4)(x+3)(x-2)(x+1)\right]$
14. If $x - 1$ and $x + 2$ are factors of $x^4 + ax^3 + bx^2 - 8x + 4$, find a and b. [$a=1,\ b=-3$]
15. List all possible rational zeros of $P(x) = 3x^3 - 5x^2 + 2x - 6$. $\left[\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{3}, \pm\frac{2}{3}\right]$
16. Find all rational zeros of $x^3 - 2x^2 - 5x + 6$. $[1, -2, 3]$
17. Find rational zeros of $2x^3 + 3x^2 - 8x + 3$. $\left[1, -3, \frac{1}{2}\right]$

18. Find all rational zeros of $4x^4 - 4x^3 - 11x^2 + 6x + 9$. $\left[\pm\frac{3}{2} \text{ and } 2 \text{ complex numbers}\right]$

19. Given 2 is a zero of $x^3 - 4x^2 + x + 6$, find all zeros. $[2,-1,3]$

20. Solve $6x^3 + 5x^2 - 17x - 6 = 0$ $\left[-\frac{3}{2},\frac{2}{3},2\right]$

21. Given $x^4 + 2x^3 - 13x^2 + 4x - 30;\ x = 3$ is a root. Find all real roots. $[-5,3;2\ complex]$

22. Given $x^4 + 2x^3 + 8x^2 + 16;\ x = -3$ is a root. Find all real roots. $[-2(\ multiplicy 2\ );2\ complex]$

23. Write a polynomial function of least degree with integer coefficients with zeros:
$x = -\frac{3}{5}$ and $1 + \sqrt{10}$ $\left[p(x) = 5x^3 - 7x^2 - 51x - 27\right]$

24. Write a polynomial function of least degree with integer coefficients with zeros:
$x = -2 + \sqrt{7}$ and $2\sqrt{2}$ $\left[p(x) = x^4 + 4x^3 - 11x^2 - 32x + 24\right]$

25. Given $p(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12$ find the value of $p(x)$ when $x = -4,-1,\text{and } 2$ $[8,-28,62]$

26. Given $p(x) = 2x^3 + 4ax^2 - 2x + 3a$ find a so $p(2) = 7$ $[a = -1]$

27. Remainder when $P(x)$ divided by $x - 1$ is 4, by $x + 2$ is -2.
Find remainder by $(x - 1)(x + 2)$. *[2x+2]*

28. When $x^3 + ax^2 + bx + 5$ divided by $x - 2$ remainder 15, by $x + 1$ remainder 2.
Find a, b. $[a = 3, b = -5]$

29. The polynomial $P(x)$ leaves a remainder of 4 when divided by $x - 2$ and a remainder of 1 when divided by $x + 1$. Find the remainder when $P(x)$ is divided by $(x - 2)(x + 1)$. $[x + 2]$

30. Find the remainder when $p(x) = x^4 - 5x^3 + 7x^2 - 4x + 1$ is divided by $(x-2)$ and $(x-3)$. Then find the remainder when $p(x)$ is divided by $x^2 - 5x + 6$ $[x - 5]$

31. $p(x) = x^4 - 3x^3 - 7x^2 + 9x + k$ has $(x+3)$ as a factor. Find the value of k and use it to factor $p(x)$ into linear factors. $[k = 72]$

32. If a polynomial is divided by x^2-4x+3, the remainder is $2x+5$. What is the remainder when is divided by $x-3$? What is the remainder when it is divided by $x-1$? [11,7]

33. $p(x)=ax^3+bx^2+cx+d$ has integer coefficients. It is known that $x=1$ and $x=-1$ are zeros of the polynomial. Additionally, when $p(x)$ is divided by $(x-3)$, the remainder is 16, and when divided by x, the remainder is - 4 . Find the values of *a, b, c* and *d*. *[a= 2, b=2 c=-2 d =-4]*

1.6 – Polynomial Graphs and Rational Functions

Topic: End Behavior, Asymptotes and Holes

Review

1. Write a polynomial function of least degree with integer coefficients with zeros:
   $x = 3 - \sqrt{5}$ and $-2\sqrt{3}$ $\quad [p(x) = x^4 - 6x^3 - 8x^2 + 72x - 48]$
2. Given $p(x) = 3x^4 - 2x^3 + 4x^2 - 5x + 6$ find the value of $p(x)$ when $x = -2, 0, \text{and } 3$ $\quad [96, 6, 216]$
3. Given $p(x) = 3x^3 + ax^2 - 5x + 2a$ find a so $p(2) = 20$ $\quad [a = 1]$
4. Remainder when $P(x)$ divided by $x - 2$ is 5, by $x + 1$ is - 3. Find remainder by $(x + 1)(x - 2)$.
   $\quad \left[R(x) = \frac{8}{3}x - \frac{1}{3}\right]$

End Behavior and Leading Term

The end behavior of a polynomial is a description of what happens as x becomes large in the positive or negative direction.
To describe end behavior the following notation is used:
$x \to \infty$ means "*x becomes large in the positive direction*"
$x \to -\infty$ means "*x becomes large in the negative direction*"

For any polynomial the end behavior is determined by the term that contains the ***highest power*** of x, because when x is large, the other terms are relatively insignificant.

polynomial with even degree $p(x) = ax^{2n} + \ldots$

if positive leading coefficient $a > 0$:
- $y \to \infty$ as $x \to \infty$
- $y \to \infty$ as $x \to -\infty$

if negative leading coefficient $a < 0$:
- $y \to -\infty$ as $x \to \infty$
- $y \to -\infty$ as $x \to -\infty$

polynomial with odd degree $p(x) = ax^{2n+1} + \ldots$

if positive leading coefficient $a > 0$:
- $y \to \infty$ as $x \to \infty$
- $y \to -\infty$ as $x \to -\infty$

if negative leading coefficient $a < 0$:
- $y \to -\infty$ as $x \to \infty$
- $y \to \infty$ as $x \to -\infty$

Example 1:

Describe the end behavior of $f(x) = -5x^3 + x^2 + 4x - 1$

Solution:

We have a negative leading coefficient (-5) and odd degree then:

$y \to -\infty$ as $x \to \infty$; $y \to \infty$ as $x \to -\infty$

Example 2:

Describe the end behavior of $f(x) = 6x^4 + x^3 - 2x + 3$

Solution:

We have a positive leading coefficient (6) and even degree then:

$y \to \infty$ as $x \to \infty$

$y \to \infty$ as $x \to -\infty$

Graphing Polynomial Functions

Step 1: Identify the degree and leading coefficient (end behavior)

Step 2: Find the *y*-intercept (set $x = 0$)

Step 3: Find *x*-intercepts (set $y = 0$)

Step 4: Graph near each zero.

For a zero at $x = c$ with multiplicity m:

- If m is odd → graph crosses the *x*-axis like a line
- If m is even → graph touches axis and turns back like a parabola

Check sign of $p(x)$ just left and right of c to know if graph is above or below axis.

Step 5: Find local extrema (turning points)

A polynomial of degree n has at most $n - 1$ turning points.

Between two consecutive x-intercepts, there is at least one local max/min.

Use symmetry if even/odd function (even: symmetric y-axis; odd: symmetric origin).

Test a few points between zeros to guess turning point location.

Step 6: Use the Intermediate Value Theorem (*IVT*)

Intermediate Value Theorem:

If $p(a)$ and $p(b)$ have opposite signs, there is at least one root between a and b.

- Pick integer or simple rational test points
- Narrow down where sign changes occur
- Works even when polynomial does not factor well

Step 7: Sketch the graph

- Plot x-intercepts (mark crossing vs touching)
- Plot y-intercept
- Draw end behavior arrows
- Connect points with smooth curve respecting turning points
- Use IVT to check missing sign changes

Example 3:

Graph $p(x)=(x+2)(x-1)^2$

Solution:

Step 1: degree 3, leading positive. Then
as $x\to\infty,\ y\to\infty$; as $x\to-\infty,\ y\to-\infty$

Step 2: $p(0)=(0+2)(0-1)^2=2$

Step 3: $x=-2, x=1$ $(multiplicity\ 2)$

Step 4:

$x<-2;$ $\quad p(-3)=-16$

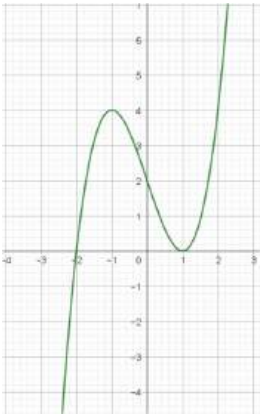
$-2<x<1;$ $\quad p(0)=2$

$x>1;$ $\quad p(2)=4$

Step 5: There are at most 2 turning points. One near $x=1$ and other between -2 and 1.

Step 6: Sketch:

1. starting bottom left,
2. cross at -2 upward,
3. rise to max between -2 and 1
4. down to touch at 1
5. then up right



Example 4:

Graph $p(x)=(x+2)\left(x^2-1\right)(x-3)$

Solution:

Step 1: degree 4, leading positive. Then
as $x\to\infty,\ y\to\infty$; as $x\to-\infty,\ y\to\infty$

Step 2: $p(0)=(0+2)\left(0^2-1\right)(0-3)=6$

Step 3: $x=-2, x=-1, x=1,\ x=3$

Step 4:

$x<-3;$ $\quad p(-3)=48$

$-2<x<-1;$ $\quad p\left(-\frac{3}{2}\right)\approx -3$

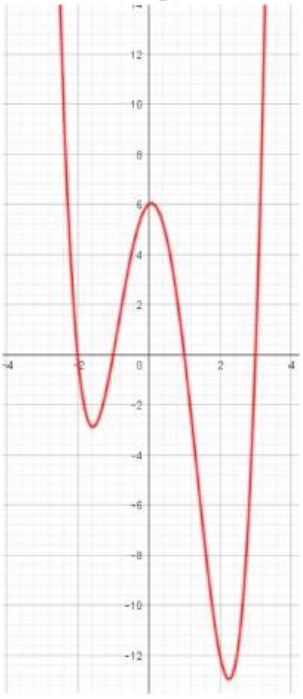
$-1<x<1;$ $\quad p(0)=6$

$1<x<3;$ $\quad p(2)=-12$

$x>3;$ $\quad p(4)=90$

Step 5: There are at most 3 turning points.
One between -2 and -1. One between -1 and 1. One between 1 and 3.
Step 6: Sketch:

1. from up left cross at -2 downward,
2. down to min between -2 and -1
3. rise to max between -1 and 1
4. down to touch at 1
5. down to min between 1 and 3
6. rise up after 3



Example 5:

Prove that $p(x)=3x^5-4x^2-3$ has a root in the interval $[0,2]$

Solution:

$p(0)=3(0)^5-4(0)^2-3=-3$

$p(2)=3(2)^5-4(2)^2-3=96-16-3=77$

Since $p(0)<0$ and $p(2)>0$ by the Intermediate Value Theorem there must be a root in $[0,2]$

Rational Functions

A rational function is defined by an expression that is quotient of two polynomials, with denominator not zero: $f(x)=\dfrac{P(x)}{Q(x)},\ Q(x)\neq 0$

where $P(x)$ and $Q(x)$ are polynomials.

The domain of $f(x)$ is all real numbers except where $Q(x)=0$.

Example 6:

Find the domain of $f(x)=\dfrac{3x+2}{x-3}$

Solution:

Denominator must be different from zero: $x-3\neq 0$.

Domain: $(-\infty,3)\cup(3,\infty)$

Example 7:

Find the domain of $f(x)=\dfrac{\sqrt{x^2-4}}{x^3-6x^2-3x+18}$

Solution:

Restrictions:

Inside the square root: radicand $\geq 0 \quad \rightarrow x^2-4\geq 0 \rightarrow x\leq -2$ or $x\geq 2$

Denominator must be different from zero:

$x^3-6x^2-3x+18\neq 0$

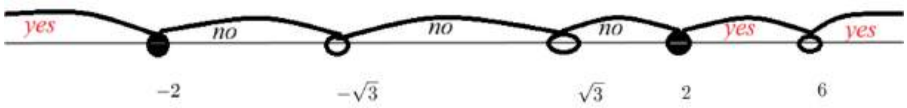
$x^2(x-6)-3(x-6)\neq 0$

$(x^2-3)(x-6)\neq 0$

$(x-\sqrt{3})(x+\sqrt{3})(x-6)\neq 0$

$x\neq \pm\sqrt{3},\ x\neq 6$

Combine Restrictions:



Domain: $(-\infty,-2]\cup[-2,6)\cup(6,\infty)$

Definition of Vertical Asymptote

The line $x=k$ is a vertical asymptote of the function $f(x)=\frac{P(x)}{Q(x)}$ if:

1. $Q(k)=0$ and $P(k)\neq 0$ and
2. as x approaches point k from the right or left $f(x)\to\pm\infty$ (the function grows without bound)

Near $x = k$, the function is undefined and the graph shoots up or down without bound, approaching a vertical line.
Graphically, the curve approaches the vertical line $x = k$ but never touches or crosses.

Example 8:

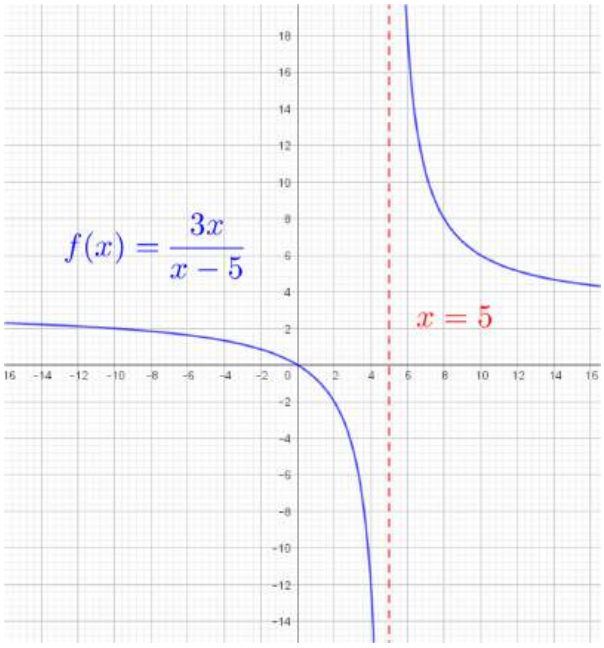
Find the vertical asymptote of $f(x)=\frac{3x}{x-5}$.

Solution:

Domain: $x-5\neq 0\to x\neq 5$.

As x approaches 5 the denominator approaches zero and $f(x)$ becomes unbounded.

Therefore $x = 5$ is a vertical asymptote.



Example 9:

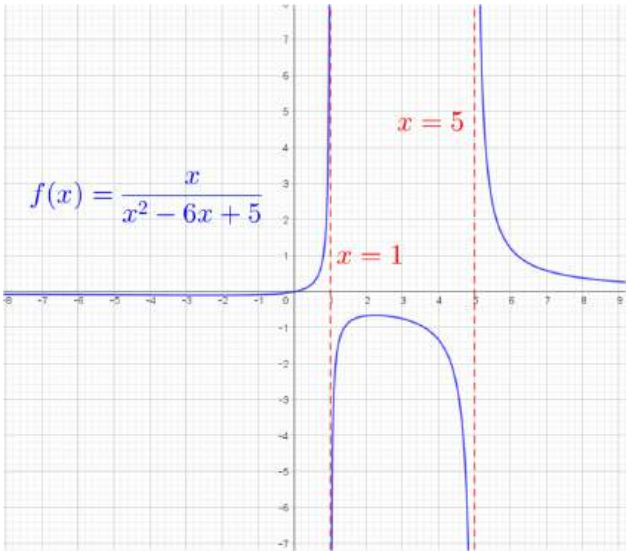
Find the vertical asymptote(s) of $f(x)=\frac{x}{x^2-3x+2}$.

Solution:

Domain: $x^2-3x+2\neq 0\to(x-1)(x-2)\neq 0\to x\neq 1,\ x\neq 2$.

As x approaches either 1 or 2 the denominator approaches zero and $f(x)$ becomes unbounded.

Therefore $x = 1$ and $x = 2$ are a vertical asymptotes.



Definition of a Hole

A **hole** (removable discontinuity) occurs in a rational function $f(x) = \frac{P(x)}{Q(x)}$ when a factor cancels completely from the numerator and denominator.

1. at $x = k$ both $Q(k) = 0$ and $P(k) = 0$
2. After canceling the common factor, the simplified function is defined at $x = k$, but the original is not.

Difference Vertical Asymptote vs. Hole

Vertical Asymptote:
the function is not defined at $x = k$, and as $x \to k$, $f(x) \to \pm\infty$ (unbounded).

Hole:
the function is not defined at $x = k$, but as $x \to k$, $f(x)$ approaches a finite number. The graph has an open circle at $(k, f(k))$ where $f(k)$ is that finite value.

Example 10:

Find the vertical asymptote(s) and hole(s) of $f(x) = \frac{x+1}{x^2 - 1}$.

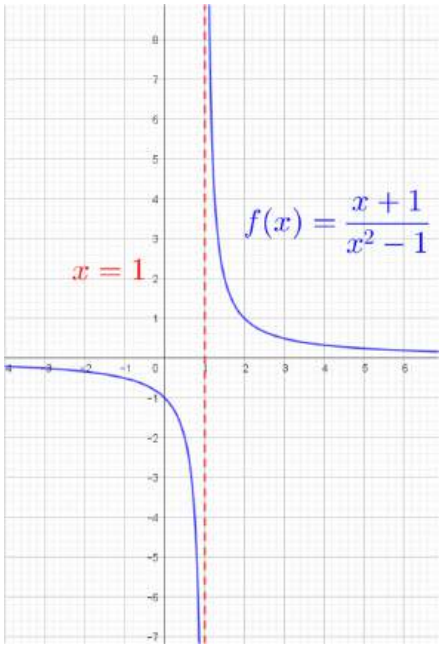
Solution:

Domain: $x^2 - 1 \neq 0 \to (x-1)(x+1) \neq 0 \to x \neq 1,\ x \neq -1$

Simplify: $f(x)=\dfrac{(x+1)}{(x+1)(x-1)}=\dfrac{1}{x-1}$

Value of $f(x)$ as x approaches 1 is unbounded. Therefore $x = 1$ is a vertical asymptote.

Value of $f(x)$ at $x = -1$: $f(-1)=\dfrac{1}{(-1)-1}=-\dfrac{1}{2}$. Therefore $x = -1$ is a hole.



Definition of Horizontal Asymptote

A **horizontal asymptote** is a horizontal line $y = b$ that the graph of a function approaches as x moves far to the right ($x \to +\infty$) or far to the left ($x \to -\infty$).

In other words: as x becomes very large (positive or negative), the function values get closer and closer to b, but may never actually reach it.

Graphically: the curve flattens out toward a horizontal line at the far left and far right ends.

3 CASES for Horizontal Asymptotes of Rational Functions

For a rational function $f(x)=\frac{P(x)}{Q(x)}=\frac{a_n x^n+\ldots}{b_m x^m+\ldots}$

where n = degree of numerator, m = degree of denominator.

CASE 1: Degree of numerator < Degree of denominator ($n < m$)

Horizontal asymptote:

$y = 0$

The denominator grows faster than the numerator when x is huge, so the fraction gets very close to zero.

CASE 2: Degree of numerator = Degree of denominator ($n = m$)

Horizontal asymptote:

ratio of leading coefficients: $y=\frac{a_n}{b_m}$

When x is huge, the lower-degree terms become insignificant, and the function behaves like $\frac{a_n}{b_m}$ **Case 3:**

CASE 3: Degree of numerator > Degree of denominator ($n > m$)

Horizontal asymptote:

None (but there may be an oblique/slant asymptote — not covered here)

The numerator grows faster than the denominator, so $f(x)$ becomes arbitrarily large as x grows ;no horizontal line is approached.

Example 11:

Find vertical asymptote(s), hole(s) and horizontal asymptote of $f(x)=\frac{2x^2-5x-3}{x^2-9}$.

Solution:

Factorize: $f(x)=\frac{2x^2-5x-3}{x^2-9}=\frac{(2x+1)(x-3)}{(x-3)(x+3)}$

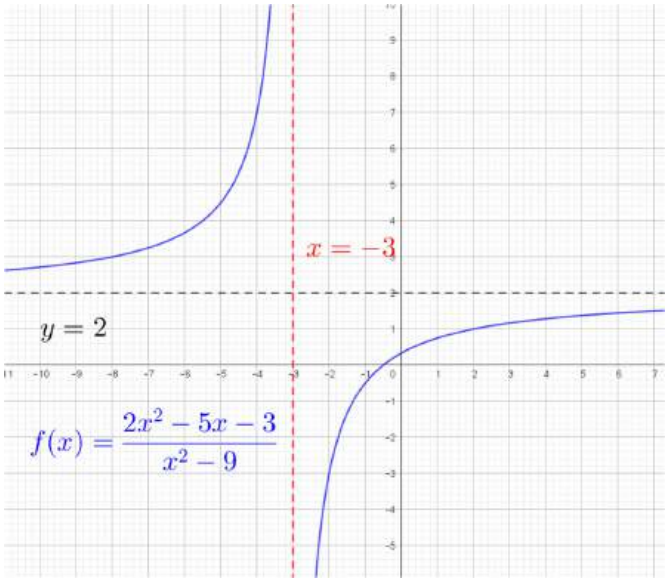
Domain: $(x-3)(x+3)\neq 0 \rightarrow x\neq 3,\ x\neq -3$

Simplify: $f(x)=\frac{2x^2-5x-3}{x^2-9}=\frac{(2x+1)(x-3)}{(x-3)(x+3)}=\frac{2x+1}{x+3}$

Value of $f(x)$ as x approaches -3 is unbounded. Therefore x =- *3* is a vertical asymptote.

Value of $f(x)$ at x $f(3)=\frac{2(3)+1}{(3)+3}=\frac{7}{6}$ -*3*: . Therefore x = *3* is a hole.

Degree numerator = degree denominator $\rightarrow y=\frac{2}{1}=2$ is an horizontal asymptote



Miscellaneous Problems

Example 12:

Decide if each statement is true or false. Explain briefly.

a) If a rational function has a horizontal asymptote at $y = 3$, it can never cross that asymptote.

b) A rational function can have more than two vertical asymptotes.

c) If the degree of the numerator is greater than the degree of the denominator, the function has no horizontal asymptote.

d) A hole occurs when the numerator and denominator share a common factor.

Solution:

a) False; horizontal asymptotes describe end behavior; the graph can cross them in the middle.

b) True; for example: $\frac{1}{(x-1)(x+2)(x-5)}$ has three vertical asymptotes (VAs).

c) True; in precalculus, no HA (oblique may exist).

d) True; provided the factor completely cancels (no remaining zero in denominator).

Example 13:

Without graphing, determine which function has a larger horizontal asymptote and explain why.

$$f(x) = \frac{5x^2 + 2x - 1}{2x^2 + 7},\ g(x) = \frac{5x^2 + 2x - 1}{3x^2 + 4}$$

Solution:

$f(x)$ has larger horizontal asymptote (HA) because denominator's leading coefficient is smaller.

$$\frac{5}{2} > \frac{5}{3}$$

Example 14:

A company manufactures x units of a product. The total cost is $C(x)=5000+20x$ dollars. The average cost per unit is $A(x)=\frac{C(x)}{x}$.

a) Write $A(x)$ as a rational function.
b) Find the vertical asymptote and explain its meaning.
c) Find the horizontal asymptote and explain its meaning.
d) If the company wants the average cost to be less than \$25, how many units must they produce?

Solution:

a) $A(x)=\frac{C(x)}{x}=\frac{5000+20x}{x}=\frac{5000}{x}+20$

b) VA at $x=0$; cannot produce zero units (undefined).

c) HA at $y=20$; as production increases, average cost approaches \$20 per unit.

d) Solve:

$$\frac{5000+20x}{x}=$$

$$\frac{5000}{x}+20<25$$

$$\frac{5000}{x}<5$$

$$x>1000$$

Example 15:

Find the value of k such that the rational function $f(x)=\frac{2x^2-kx+3}{x^2-4}$ has a horizontal asymptote at $y=2$ and a hole at $x=2$.

Solution:

Factorize: $f(x)=\frac{2x^2-kx+3}{x^2-4}=\frac{2x^2-kx+3}{(x-2)(x+2)}$

Hole at $x=2$ means also the numerator has $(x-2)$ as a factor

$$\rightarrow 2(2)^2-k(2)+3=0$$

$$\rightarrow 8-2k+3=0$$

$$\rightarrow k=\frac{11}{2}$$

Example 16:

Find $f(x)=\frac{P(x)}{Q(x)}$ such that: vertical asymptote at $x=3$, horizontal asymptote at $y=0$, passes through point (1,2)

Solution:

1. HA: $y = 0 \rightarrow$ degree denominator > degree numerator
2. VA: $x = 3 \rightarrow Q(x) = x - 3$ and $\rightarrow P(x) = k \ (constant)$
3. $\rightarrow f(x) = \frac{k}{x-3}$ using point (1,2) $\rightarrow 2 = \frac{k}{(1)-3} \rightarrow k = -4$
4. $f(x) = -\frac{4}{x-3}$

Example 17:

Find $f(x) = \frac{P(x)}{Q(x)}$ such that: VA: $x = -2$ and at $x = 4$, HA: at $y = 3$,

x-intercept at $x = 1$ and y-intercept at $y = -\frac{3}{8}$

Solution:

1. HA: $y = 3 \rightarrow$ degree denominator = degree numerator
2. VA: $x = -2$ and $x = 4 \rightarrow Q(x) = (x+2)(x-4) = x^2 - 2x - 8$
3. x-intercept at $x = 1$ and degree denominator = degree numerator $\rightarrow P(x) = a(x-1)^2$
4. so we have: $f(x) = \frac{P(x)}{Q(x)} = \frac{a(x-1)^2}{x^2 - 2x - 8}$ using y-intercept $\left(0, -\frac{3}{8}\right)$:

   $-\frac{3}{8} = \frac{a(0-1)^2}{(0)^2 - 2(0) - 8} \rightarrow -\frac{3}{8} = \frac{a}{-8} \rightarrow a = 3$
5. $f(x) = \frac{3(x-1)^2}{x^2 - 2x - 8}$

Example 18:

Find $f(x) = \frac{P(x)}{Q(x)}$ such that: VA: $x = -3$ and $x = 2$; HA: $y = 0$; hole at $x = 1$; passes through $\left(0, -\frac{2}{3}\right)$

Solution:

1. VA: $x = -3$ and $x = 2$ and hole at $x = 1 \rightarrow Q(x) = a(x-2)(x-1)(x+3)$
2. HA: $y = 0 \rightarrow$ degree numerator < degree denominator and hole at $x = 1 \rightarrow P(x) = b(x-1)$
3. therefore $f(x) = \frac{b(x-1)}{a(x-2)(x-1)(x+3)}$ using $\left(0, -\frac{2}{3}\right)$

   $-\frac{2}{3} = \frac{b(0-1)}{a(0-2)(0-1)(0+3)} \rightarrow -\frac{2}{3} = \frac{-b}{6a} \rightarrow 12a = 3b \rightarrow b = 4a$

   Let $a = 1$ then $b = 4$ then $f(x) = \frac{4(x-1)}{(x-1)(x+3)(x-2)}$

Practice Problems

1. Describe the end behavior of $f(x) = -2x^5 + 3x^3 - x + 7$ $\left[x \to -\infty, y \to \infty; x \to \infty, y \to -\infty\right]$

2. Describe the end behavior of $f(x) = 4x^6 - 2x^3 - 1$ $\left[x \to \pm\infty, y \to \infty\right]$

3. Describe the end behavior of $f(x) = -3x^4 + 2x^2 - x + 3$ $\left[x \to \pm\infty, y \to -\infty\right]$

4. Find a polynomial with odd degree, positive leading coefficient, and as $x \to -\infty, y \to -\infty$

$\left[f(x) = x^3\text{; any positive odd degree works}\right]$

5. Find a polynomial with even degree, negative leading coefficient, and as $x \to \infty, y \to -\infty$

$\left[f(x) = -x^2\right]$

6. If $p(x) = a_n x^n + \ldots$ has end behavior: as $x \to -\infty, y \to \infty$; and as $x \to \infty, y \to -\infty$ what can you say about a_n a and n? $\left[n \text{ is odd, } a_n < 0\right]$

7. Graph $f(x) = (x+2)^2(x-1)$. List end behavior, intercepts, and behavior at zeros.

[End: left down, right up; y-intercept (0, -4); x-intercepts: -2 (touch), 1 (cross)]

8. Graph $f(x) = -(x-3)(x+1)^3$ .List end behavior, intercepts, and behavior at zeros.

[End: left up, right down; y-intercept (0, 3); x-intercepts: 3 (cross), -1 (cross, flatter)]

9. Graph $f(x) = x^4 - 4x^2$. Find all intercepts and turning points.

[y-intercept (0,0); x-intercepts: -2, 0, 2; turning points at $x \approx$ -1.41, 0, 1.41]

10. Prove that $f(x) = x^3 - 4x + 1$ has a root between 0 and 1. $\left[f(0) > 0, f(1) < 0\right]$

11. Prove that $f(x) = x^5 + 2x - 1$ has a root between 0 and 1. $\left[f(0) >< 0, f(1) > 0\right]$

12. How many turning points can $f(x) = x^5 - 3x^2 + 2$ have at most? [4]

13. Find domain of $f(x) = \dfrac{2x+1}{x^2-9}$ [All reals except x ≠ 3, -3]

14. Find domain of $f(x) = \dfrac{\sqrt{x+5}}{x^2-4}$ [-5, -2) ∪ (-2, 2) ∪ (2, ∞)

15. Find domain o $f(x) = \dfrac{3x}{\sqrt{x^2-1}}$ [(-∞, -1) ∪ (1, ∞)]

16. Find domain of $f(x)=\dfrac{\sqrt{4-x}}{x^2-5x+6}$ [(-∞, 2) ∪ (2, 3) ∪ (3, 4]

17. Find domain of $f(x)=\dfrac{x-2}{\sqrt{9-x^2}}$ [(-3, 3)]

18. Find domain of $f(x)=\dfrac{\sqrt{x^2-4}}{x^2-x-6}$ [(-∞, -2] ∪ [2, 3) ∪ (3, ∞)]

19. Find VA of $f(x)=\dfrac{2x}{x-4}$ [$x=4$]

20. Find VA of $f(x)=\dfrac{x^2-1}{x^2-5x+6}$ [$x = 2, x = 3$]

21. Find VA of $f(x)=\dfrac{3x+1}{x^2+4}$ [None]

22. Find VA and holes of $f(x)=\dfrac{x^2-4}{x^2-4x+4}$ [VA:None; hole at $x=2$)]

23. Find holes and VA of $f(x)=\dfrac{x^2-9}{x^2-6x+9}$ [VA:$x = 3$; hole at $x=-3$)]

24. Find holes and VA of $f(x)=\dfrac{x^2+2x-8}{x^2+8x+16}$ [Hole at $x = -4$; VA at $x = 1$]

25. Find holes and VA of $f(x)=\dfrac{x^2-1}{x^2+x-2}$ [Hole at $x = 1$; VA at $x =-2$]

26. Find holes and VA of $f(x)=\dfrac{x^3-4x}{x^2-4}$ [Hole at $x = 2$; VA at $x =-2$]

27. Find holes and VA of $f(x)=\dfrac{x^2-5x+6}{x^2-9}$ [Hole at $x = 3$; VA at $x =-3$]

28. Find holes and VA of $f(x)=\dfrac{x^3-3x+2}{x^2+2x-3}$ [Hole at $x = 1$; VA at $x =-3$]

29. Find HA of $f(x)=\dfrac{7x^2-4x+1}{x^2-9}$. [$y = 7$]

30. Find HA $f(x)=\dfrac{5}{x-9}$ [$y = 0$]

31. Find HA of $f(x)=\dfrac{3x^4-2x}{2x^4+x^3+2}$ $\left[y=\dfrac{3}{2}\right]$

32. Find HA of $f(x)=\dfrac{\sqrt{4x^2-2}}{8x-5}$ $\left[y=\dfrac{1}{4}\right]$

33. Find k so that $f(x)=\dfrac{x^2-k}{x-2}$ has a hole at $x = 2$. $[k=4]$

40. Find k so that $f(x)=\dfrac{3x^2+kx}{x^2-9}$ has horizontal asymptote $y = 3$ and no holes. $[k=\pm 9]$

41. Find $f(x)=\dfrac{P(x)}{Q(x)}$ with VA: at $x = -1$, $x = 3$; HA: at $y = 0$; x-intercept at $x = 2$; y-intercept at $y=\dfrac{2}{3}$.

$\left[f(x)=\dfrac{x-2}{x^2-2x-3}\right]$

42. Find $f(x)=\dfrac{P(x)}{Q(x)}$ with VA: at $x = -4$, $x = 1$; HA at $y = 0$; x-intercept at $x = -2$; y-intercept at $y=\dfrac{1}{2}$.

$\left[f(x)=\dfrac{2(x+2)}{(x^2+3x-3)}\right]$

43. Find $f(x)=\dfrac{P(x)}{Q(x)}$ such that: vertical asymptote at $x = 7$, horizontal asymptote at $y = 0$, passes through point $\left(-1,\dfrac{5}{8}\right)$ $\left[f(x)=\dfrac{5}{7-x}\right]$

44. Find $f(x)=\dfrac{P(x)}{Q(x)}$ such that: VA: $x = 3$ and at $x = -4$, HA: at $y = 3$, x-intercept at $x = 2$ and y-intercept at $y=-\dfrac{4}{3}$ $\left[f(x)=\dfrac{4(x-2)^2}{x^2+x-12}\right]$

45. Find $f(x)=\dfrac{P(x)}{Q(x)}$ such that: VA: $x = -2$ and $x = 2$; HA: $y = 0$; hole at $x = 2$; passes through $(0,-5)$

$\left[f(x)=\dfrac{5(x-2)}{(x-1)(x+1)(x-2)}\right]$

CHAPTER 1 – Review

1. Find the value(s) of a such that the domain of $f(x)=\dfrac{\sqrt{x+a}}{x-3}$ is $[-2,3)\cup(3,\infty)$. $[a=2]$

2. Find the domain of $f(x)=\dfrac{\sqrt{x^2-9}}{x^2-4x+3}$. $[(-\infty,-3)\cup(3,\infty)]$

3. If the domain of $f(x)=\dfrac{\sqrt{x-a}}{x^2-9}$ is $[2,3)\cup(3,8)$, find a $[a=2]$

4. If the domain of $f(x)=\dfrac{\sqrt{x-a}}{x^2-bx+6}$ is $[4,6)\cup(6,8)$ and the graph passes through (0, -2) find a and b. $[a=4,b=10]$

5. Find the value of a so that the function $f(x)=3x+a$ through the point (4, 14). $[a=2]$

6. Find the value(s) of a such that $f(x)=\dfrac{x^2-a}{x-2}$ has a hole at $x=2$. $[a=4]$

7. Given $f(x)=\dfrac{3x+a}{x-5}$ such that $f(2)=4$, find a. $[a=2]$

8. Given $f(x)=\dfrac{2x^2+ax}{x^2-4}$ such that $f(1)=-1$, find a. $[a=-3]$

9. Given $f(x)=\dfrac{x^2+a}{x+1}$ such that $f(3)=5$, find a. $[a=11]$

10. Given $y=\dfrac{3}{kx^2+6x+3}$ find the value of k so the domain of the function is $x\neq -1$ *[k = 3]*

11. Given $y=\dfrac{2}{ax^2+2x+5}$ find the value of a so the domain of the function is $x\neq 5$ $\left[a=-\dfrac{3}{5}\right]$

12. Find the range of $f(x)=\dfrac{2x-3}{x-1}$

    $[(-\infty,2),(2,\infty)]$

13. The cost of producing 15 shirts is 210 euro, and the cost of producing 8 shirts is 126 euro. If the cost C can be described by $C(x)=mx+q$ $C=mx+q$, find:

    (i) the marginal cost [Marginal cost = 12 euro per unit]

    (ii) the cost function $C(x)$ $[C(x)=12x+30]$

14. A company finds that the total cost of producing x units is $C(x)=15x+200$ and the revenue is $R(x)=25x$. [Break-even point: $x=20$ units]

15. A taxi service charges a flat base fee of €3.50 plus €0.50 per kilometer traveled.

    (i) Let x be the distance in km. Write an expression for the cost respect to $[C(x)=3.50+0.5x]$

    (ii) Find the cost of a 12 km trip. [9.50]

    (iii) If a trip costs €12.00, how many kilometers did the passenger travel? [17]

16. Express the area of a square as a function of its diagonal d. State the domain. $\left[A=\dfrac{d^2}{2};d>0\right]$

17. A newspaper ad costs \$10 plus \$0.30 for each word after the first 15 words.

    (i) Write an expression for the cost $C(w)$ for an ad with w words. $[C(w)=10+0.3(w-15)]$

    (ii) John paid \$19.00 for his ad. How many words were in it? [45]

18. Find the value of a that makes $f(x)$ continuous at $x = 3$:

$$f(x)=\begin{cases} x^2-5 & \text{if } x\le 3 \\ ax+1 & \text{if } x>3 \end{cases}$$ $[a=1]$

19. Given $f(x)=x^2-16$ find $\dfrac{f\left(\frac{1}{4}+h\right)-f\left(\frac{1}{4}\right)}{h}$ $\left[\frac{1}{2}+h\right]$

20. The function $f(x)=kx^2+3$ has an average rate of change of 7 on $[2,5]$. Find k. $\left[\frac{1}{14}\right]$

21. The area under $f(x)=4x+1$ on $[0,a]$ is 18. The average value of $f(x)$ on the same interval is m.

Find a and m. $\left[a=\frac{11}{4}, m=\frac{72}{11}\right]$

22. The area under $f(x)=x^2+3x$ on $[2,c]$ is $\frac{141}{2}$. The average value of f on the same interval is m.

Given that the average rate of change on $[2,c]$ is 10 , find c and m. $\left[c=5, m=\frac{47}{2}\right]$

23. Given $f(x)=\dfrac{4x}{x-3}, g(x)=\dfrac{x-5}{x+2}$, find $(f\circ g)(3)$ $\left[\frac{8}{17}\right]$

24. Given $f(g(x))=30-8x$ and $g(x)=7-3x$, find $f(x+2)$ $\left[\frac{8x+50}{3}\right]$

25. Given $(f\circ g)(x)=\dfrac{1}{x^2+6x+10}$ and $g(x)=x+3$, find $f(x)$ $\left[\frac{1}{x^2+1}\right]$

26. Given $f(g(x))=\dfrac{3x}{x-2}$ and $g(x)=\dfrac{2}{x}$, find $f(x+1)$ $\left[-\frac{3}{x}\right]$

27. Find the inverse of $f(x)=\dfrac{\sqrt[3]{5x}-3}{4}$ $\left[\frac{(4x+3)^3}{5}\right]$

28. Find the inverse of $f(x)=\dfrac{3x+2}{x-5}$ then find the domain and range of both $f(x)$ and $f^{-1}(x)$ $\left[\frac{5x+2}{x-3}\right]$

domain $f(x): x\neq 5$, range $y\neq 3$ − − − domain $f^{-1}(x): x\neq 3$, range $y\neq 5$

29. Given $f(x)=\sqrt{x+3}$ and $g(x)=ax+b$. Find a and b if given that $(f\circ g)(1)=3$ and $g(-1)=1$

$\left[a=\frac{5}{2}, b=\frac{7}{2}\right]$

30. Given $f(x)=\sqrt{x+4}$ and $g(x)=ax+b$, find a and b if $(f\circ g)(2)=5$ and $g(3)=8$

$[a=-13, b=47]$

31. Let $f(x)=3x+2$ and $g(x)=\dfrac{1}{x-2}$. Solve the equation $f^{-1}(x)=(g\ o\ f)(x)-1$

$\left[x=\dfrac{-1\pm\sqrt{5}}{2}\right]$

32. Let $f(x)=x-2$ and $g(x)=3x+1$. Determine if $(f\ o\ g)^{-1}=g^{-1}\ o\ f^{-1}$ [yes]

33. Let $f(x)=\sqrt{x-5}$ and $g(x)=x^2+3$. The function $(f\ o\ g)(x)$ is defined for all real numbers except for the interval (a,b). Find the values of a and b. $[a=-\sqrt{2}, b=\sqrt{2}]$

34. If $f(x)=\dfrac{2x}{x+3}$ and $g(x)=f(f(x))$ find $g(g(1))$ $\left[\dfrac{8}{73}\right]$

35. Write the standard form equation of a parabola with vertex $(3,-4)$ and passing through $(5,8)$

$[y=3x^2-18x+23]$

36. Write the standard form equation of a parabola with vertex $(3,-4)$ and $x-$intercepts at $x=$ -4 and $x=0$ $\left[y=-\dfrac{5}{4}x^2-5x\right]$

37. Write the standard form equation of a parabola with vertex $(1,-6)$, axis of symmetry vertical and passing through $(-2,12)$ $[y=2x^2-4x-4]$

38. Find the value of a and b so the parabola $y=ax^2+bx-3$ passe through points (1, 2) and (- 2, 5).

[a=3,b=2]

39. Find the values of a and b so the parabola $y=ax^2+bx+4$ goes through points $(2,0)$ and $(-1,9)$

[a=1,b= - 4]

40. Write a polynomial function of least degree with integral coefficients that has the given zeros.

(a) Zeros: $\dfrac{5}{4},-3,5$ $[p(x)=4x^3-13x^2-50x+75]$

(b) Zeros: $-\dfrac{2}{5},-3+\sqrt{7}$ $[p(x)=5x^3+32x^2+22x+4]$

(c) Zeros: $-2+\sqrt{6},2+\sqrt{7},2-\sqrt{7}$ $[p(x)=x^4-21x^2-4x+6]$

(d) Zeros: $-\dfrac{5}{2},\sqrt{6}$ $[p(x)=2x^3+5x^2-12x-30]$

41. Find all the real roots and describe the multiplicity (if necessary):

(a) $x^4-13x^2+40=0$ $\left[\sqrt{5},-\sqrt{5},2\sqrt{2},-2\sqrt{2}\right]$

(b) $x^3-4x^2+4x=0$ $[0,2\ mult.\ 2]$

(c) $x^4-4x^3-10x^2=0$ $\left[0\,mult.\ 2,2+\sqrt{14},2-\sqrt{14}\right]$

(d) $x^4-8x^2+12=0$ $\left[\sqrt{6},-\sqrt{6},\sqrt{2},-\sqrt{2}\right]$

42. Factor each. One root has been given.

$x^4-5x^3-2x^2+24x=0;\ -2$ $\left[x(x-3)(x-4)(x+2)=0\right]$

$x^4-2x^3-11x^2+12x=0;\ -3$ $\left[x(x-3)(x-4)(x+2)=0\right]$

$x^4-7x^3-5x^2+75x=0;\ 5$ $\left[x(x+3)(x-5)^2=0\right]$

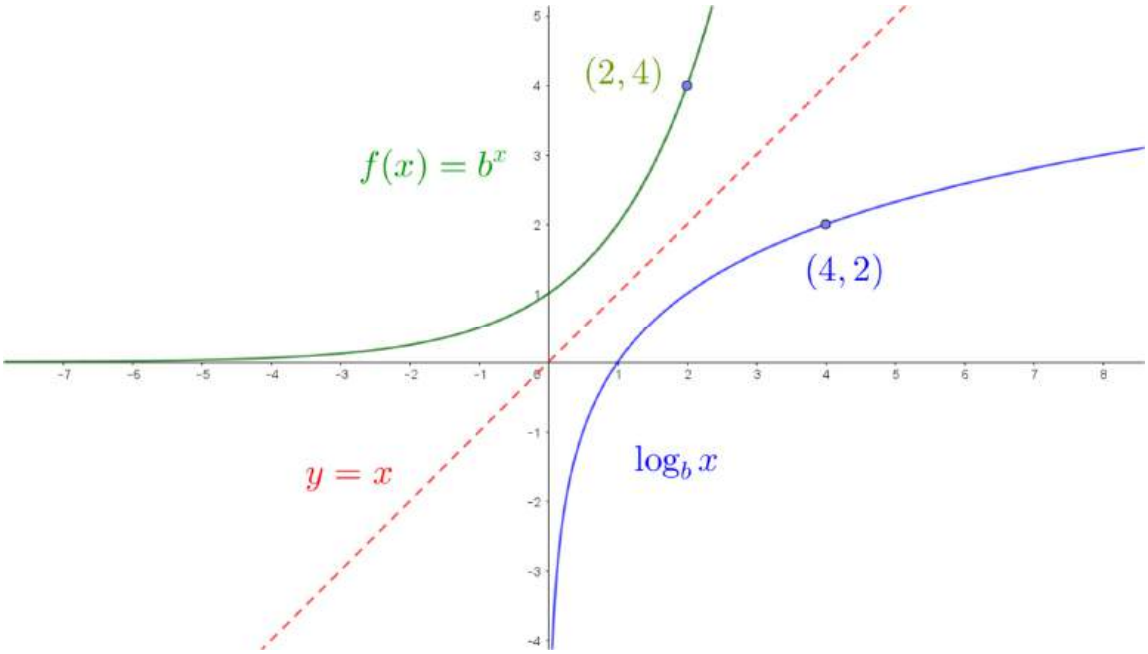
43. $(x-2)$ and $(x+4)$ are factors of x^3+ax^2+bx+c, and it leaves a remainder of 10 when divided by $(x-1)$. Find the values of *a, b* and *c*. $[a=-1,b=-14,c=24]$

44. The polynomial $p(x)=(ax+b)^3$ leaves a remainder of 27 when divided by $(x+1)$ and a remainder of 8 when divided by $(x-2)$. Find the values of *a* and *b*. $[a=1,b=4]$

45. The polynomial $p(x)=2x^3+x^2+ax+b$ leaves a remainder of 20 when divided by $(x-2)$ and is exactly divisible by $(x+3)$. Find the values of *a* and *b*. $[a=-13,b=6]$

CHAPTER 2-Exponential and Logarithms Functions

"By shortening the labours, [Napier's logarithms] doubled the life of the astronomer."-Pierre-Simon Laplace



Review

Properties of Exponents: For $a > 0$, $b > 0$, and $m, n \in \mathbb{R}$:

1. $a^m \times a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $\left(a^m\right)^n = a^{m \times n}$
4. $(a \times b)^m = a^m \times b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
6. $a^0 = 1$
7. $a^{-n} = \frac{1}{a^n}$
8. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

One-to-One Property: If $a > 0$ *and* $a \neq 1$, then: $a^m = a^n$ if and only if $m = n$

1. Evaluate: 4^{-3} $\left[\frac{1}{64}\right]$

2. Simplify: $\frac{27^{\frac{2}{3}} \times 81^{\frac{3}{4}}}{\left(3^5\right)}$ $\left[a^{\frac{4}{3}}\right]$

3. Simplify: $ab\left(a^2b - ab^2\right)$ $\left[a^3b^2 - a^2b^3\right]$

4. Simplify: $\frac{a^{\frac{2}{3}} \times a^{\frac{4}{3}}}{\left(a^{\frac{1}{3}}\right)^2}$ [1]

2. 1 Exponential Functions

Topic: exponents, natural base, graphs, growth/decay,equations

An exponential function is a function of the form: $f(x)=b^x$

where $b>0, b\neq 1$, and x is any real number.

b is called the **base**

x is called the **exponent** (the variable)

The Natural Exponential Function

The natural exponential function is: $f(x)=e^x$

where e ≈ 2.718281828... is Euler's number.

This is the most important exponential function in calculus and natural sciences.

Example 1:

Write in exponential form $\sqrt[3]{\dfrac{x^2y}{z^6}}$.

Solution:

$$\sqrt[3]{\frac{x^2y}{z^6}}=\left(\frac{x^2y}{z^6}\right)^{\frac{1}{3}}=\frac{\left(x^2y\right)^{\frac{1}{3}}}{\left(z^6\right)^{\frac{1}{3}}}=\frac{x^{\frac{2}{3}}y^{\frac{1}{3}}}{z^2}$$

Example 2:

Let $f(x)=\left(\sqrt[3]{x^2}\right)\left(\sqrt{x^3}\right)$, write in exponential form and in simplest radical form.

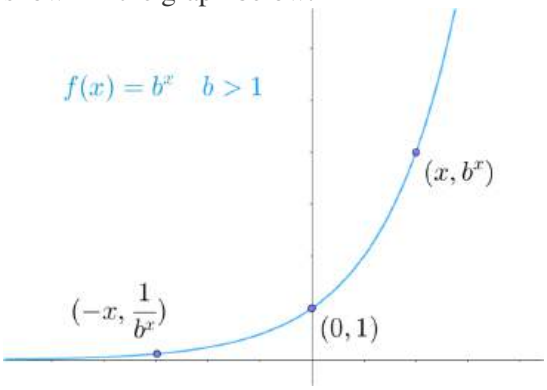
Solution:

$$f(x)=\left(\sqrt[3]{x^2}\right)\left(\sqrt{x^3}\right)=\left(x^{\frac{2}{3}}\right)\left(x^{\frac{3}{2}}\right)=x^{\frac{4+9}{6}}=x^{\frac{13}{6}}\ \text{(exponential form)}$$

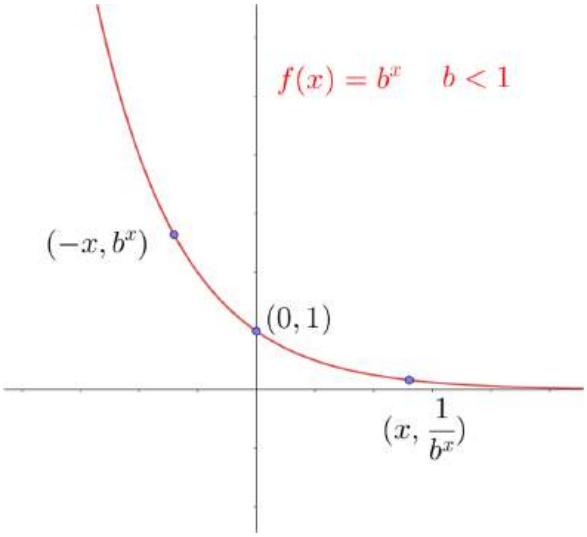
$$f(x)=\left(\sqrt[3]{x^2}\right)\left(\sqrt{x^3}\right)=x^{\frac{13}{6}}=x^{2+\frac{1}{6}}=x^2\sqrt[6]{x}\ \text{(radical form)}$$

Graphs of Exponential Functions $f(x)=b^x$

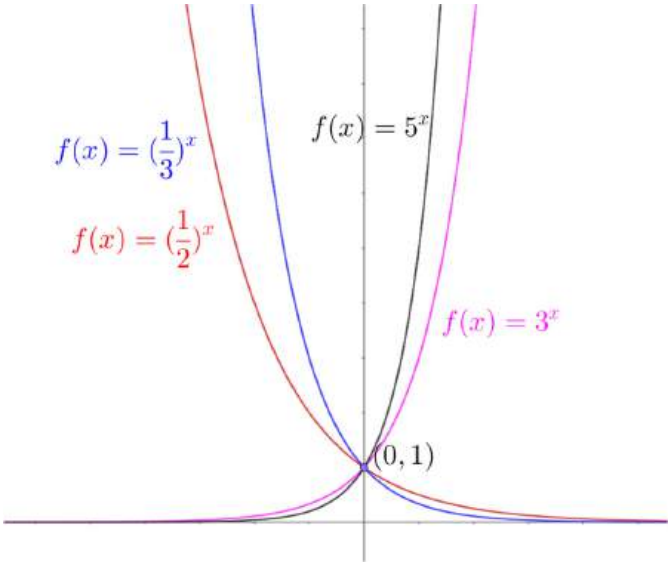
The graph of any exponential function whose base is greater than 1 has the same general form as the graph as shown in the graph below.



The graph of any exponential function whose base is between 0 and 1 has the same general form as the graph as shown in the graph below.



All exponential functions have the same characteristics. The only thing that varies is the steepness of their graphs, as we can see from the next graph.



From the graphs, we observe some characteristics of $f(x) = b^x$ either $b > 1$ or $0 < b < 1$:

1. The domain is all real numbers
2. The range is $y > 0$
3. The graph crosses the y-axis at $(0, 1)$
4. The graph approaches the x-axis but will never touches it, since y cannot be 0. This makes the x-axis a horizontal *asymptote*. An asymptote is a straight line whose perpendicular distance from a curve decreases to zero as the distance from the origin increases without limit.
5. With $b > 1$ the value of y increases as the value of x increases. (**growth**)
6. With $0 < b < 1$ the value of y increases as the value of x decreases. (**decay**)

Summary of Properties of Exponential Function

For $f(x)=b^x$ where $b > 0$, $b \neq 1$:

Domain: All real numbers $(-\infty, \infty)$
Range: $(0, \infty)$
***y*-intercept:** $(0, 1)$ because $b^0 = 1$
***x*-intercept:** None (graph never crosses x-axis)
Horizontal asymptote: $y = 0$ (the x-axis)
Continuity: Continuous for all real numbers
Monotonicity:
If $b > 1$, the function is strictly increasing
If $0 < b < 1$, the function is strictly decreasing

End Behavior

For $f(x)=b^x$:

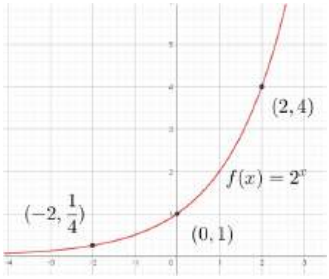
If $b > 1$:
as $x \to -\infty$, $f(x) \to 0$
as $x \to \infty$, $f(x) \to \infty$

If $0 < b < 1$:
as $x \to -\infty$, $f(x) \to \infty$
as $x \to \infty$, $f(x) \to 0$

Example 3:

Given $f(x)=2^x$. Graph $f(x)$ and show the values of $f(-2), f(0), f(2)$

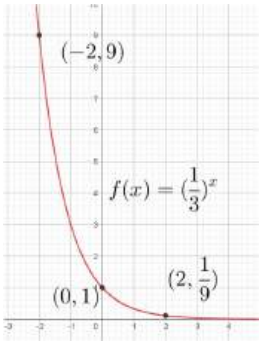
Solution:



Example 4:

Given $f(x)=\left(\frac{1}{3}\right)^x$. Graph $f(x)$ and show the values of $f(-2), f(0), f(2)$

Solution:



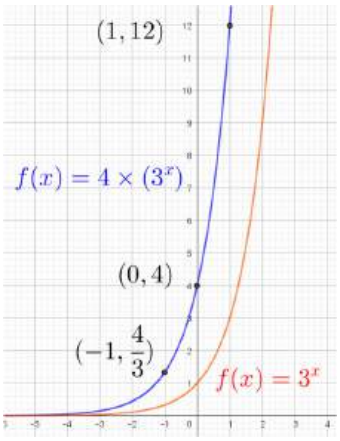
Graphs of Exponential Functions $f(x)=k(b^x)$ and $f(x)=k(b^{ax})$

In the same way if we have a function $f(x)=k\times b^x$ with $k\in\mathbb{R}$ its graph will have the same shape of the ones before but the value of y is k times the corresponding y-value of $f(x)=k\times b^x$.

Example 5:

Given $f(x)=4\times 3^x$. Graph $f(x)$ and show the values of $f(x)=4\times 3^x$

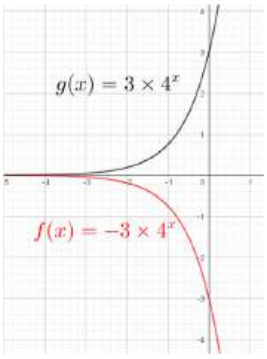
Solution:



Example 6:

Given $f(x)=-3\times 4^x$ and $g(x)=3\times 4^x$. Graph both $f(x)$ and $g(x)$

Solution:



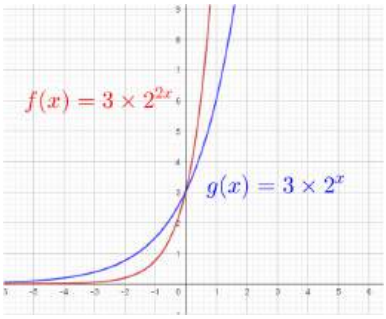
As we can see the two graphs are the reflections of each other in the x-axis.

The steepness of the graph of an exponential function is also determined by the value of the exponent. For example, if we have $f(x)=k(b^{ax})$ the graph becomes steeper as the value of a increases.

Example 7:

Given $f(x)=3\times 2^{2x}$ and $g(x)=3\times 2^x$. Graph both $f(x)$ and $g(x)$

Solution:



General Function $f(x) = k\left(b^{ax}\right) + c$

One more thing to consider is the change of the asymptote if there is a value to add; in this way the general form of an exponential function is $f(x) = k\left(b^{ax}\right) + c$.

For every value of *x*, the value of the asymptote is shifted by *c* units, either up or down according to the value of *c*, respect to the *x*-axis.
We can extend some main points to the previous list:
given $f(x) = k\left(b^{ax}\right) + c$

- The range is either $y > c$ or $y < c$ depending on the value of *k (positive/negative)*.
- The graph crosses the *y*-axis at *(0, k + c)*
- For $x = 1$ the *y* coordinate is $k \times b^a + c$
- For $x = -1$ the *y* coordinate is $\frac{k}{b^a} + c$
- For *x* infinitely large either positive or negative the graph approaches the value $f(x) = c$ but will never touches it, therefore the line $y = c$ a horizontal *asymptote*. If the horizontal asymptote is below the *x*-axis then there is also the *x*-intercept at $y = 0$.

Example 8:

Given $f(x) = 5 \times 2^{3x} + 7$ find:

(i) The value of *k, a, b* and *c*
(ii) The range of the function
(iii) The equation of the horizontal asymptote
(iv) The *y*-intercept and the possible *x*-intercept
(v) The value of *y* for $x = 1$ *and* -1
(vi) Draw its graph

Solution:

(i) The general exponential function is $f(x) = k \times b^{ax} + c$ so we have

$k = 5 \quad a = 3 \quad b = 2 \quad$ *and* $c = 7$

(ii) Since $c = 7$ the range is $y > 7$

(iii) Since $c = 7$, it follows that the equation of the horizontal asymptote is $y = 7$

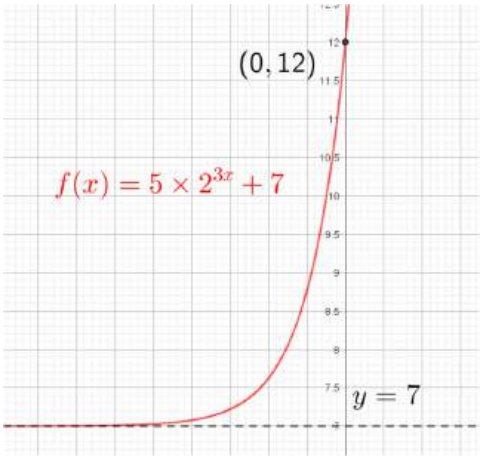
(iv) The *y*-intercept occurs when $x = 0$ so we have $y = 5 \times 1 + 7 = 12$

There is no *x*-intercept since $c = 7$ *and* $k > 0$.

(v) For $x = 1$ we have $y = 5 \times 2^3 + 7 = 5 \times 8 + 7 = 47$

For $x = -1$ we have $y = \frac{5}{2^3} + 7 = \frac{5 + 7 \times 2^3}{2^3} = \frac{61}{8}$

(vi)



Example 9:

Given $f(x) = -4 \times 2^{2x} + 64$ find:

(i) The value of *k, a, b* and *c*

(ii) The range of the function

(iii) The equation of the horizontal asymptote

(iv) The *y*-intercept and the possible *x*-intercept

(v) The value of *y* for *x = 1 and -1*

(vi) Draw its graph

Solution:

(i) The general exponential function is $f(x) = k \times a^{bx} + c$ so we have

k = -4 a = 2 b = 2 and c = 64

(ii) Since *c = 64 and k < 0* the range is *y < 64*

(iii) Since *c = 64*, it follows that the equation of the horizontal asymptote is *y = 64*

(iv) The *y*-intercept occurs when *x = 0* so we have $y = -4 + 64 = 60$

The *x*-intercept occurs when *y = 0* so we have

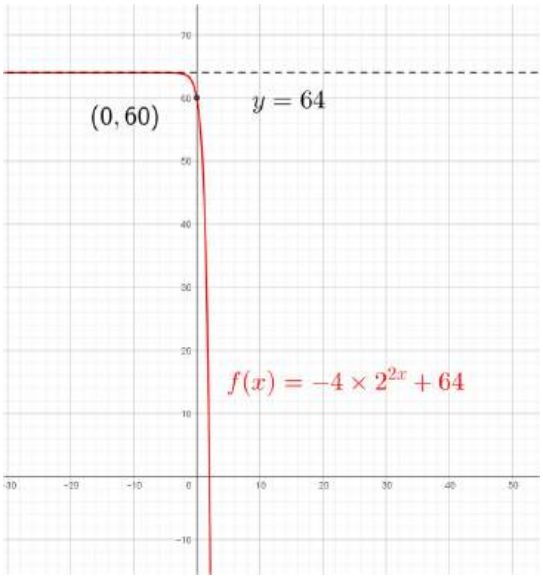
$-4 \times 2^{2x} + 64 = 0$ it follows $-4 \times 2^{2x} = -64$ dividing by – 4 it will give us $2^{2x} = \frac{-64}{-4} = 16$.

Since $16 = 2^4$ we have $2^{2x} = 2^4$ then $2x = 4$ that is $x = 2$.

(v) For *x = 1* we have $y = -4 \times 2^2 + 64 = -16 + 64 = 48$

For *x = -1* we have $y = \frac{-4}{2^2} + 64 = 63$

(vi)



Exponential Equations

Solving an exponential equations we must: make the bases on both sides of the equation exactly the same, then equalize their exponents.

Remember: if $b^x = b^y$, then $x = y$ $(b > 0, b \neq 1)$

Steps

1. Identify and find a common base
2. Substitute the new bases into the equation
3. Apply Properties of Exponents
4. Drop the bases and set the exponents equal
5. Solve the remaining equation.

Example 10:

Solve $3^{2x} = 3^8$

Solution:

Since the bases are the same it follows that $2x = 8$ then $x = 4$.

Example 11:

Solve $4^{2x} = 16^{3x+2}$

Solution:

Since $16 = 4^2$ then we have $4^{2x} = \left(4^2\right)^{3x+2}$. Applying the properties of exponents: $4^{2x} = 4^{6x+4}$.

Drop the bases and set the exponents equal: $2x = 6x + 4$. Solving for x:

$2x - 6x = 4$

$x = -1$

Example 12:

Solve $4^{x^2-2x} = 8^{1-x}$

Solution:

$4^{x^2-2x}=8^{1-x}$

$\left(2^2\right)^{x^2-2x}=\left(2^3\right)^{1-x}$

$2^{2x^2-4x}=2^{3-3x}$

$2x^2-4x=3-3x$

$2x^2-x-3=0$

$(2x-3)(x+1)=0$

$x=\frac{3}{2}, x=-1$

Example 13:

Solve $2^{(x+1)}+2^{(x-1)}=20$

Solution:

$2^{(x+1)}+2^{(x-1)}=20$ $\quad\left[2^{(x+1)}=2\times 2^x \text{ and } 2^{(x-1)}=\frac{2^x}{2}\right]$

$2\times 2^x+\frac{2^x}{2}=20$ (multiply both sides by 2)

$4\times 2^x+2^x=40$ (4+1=5)

$5\left(2^x\right)=40$ (divide by 5)

$2^x=8$

$2^x=2^3$

$x=3$

Example 14:

Solve $3^{2x}-10\times 3^x+9=0$

Solution:

Rewrite $3^{2x}-10\times 3^x+9=0$ as $\left(3^x\right)^2-10\times 3^x+9=0$. Then let $u=3^x$, it follows:

$\left(3^x\right)^2-10\times 3^x+9=u^2-10u+9=0$

Solve for u by factorization:

$u^2-10u+9=0$

$(u-1)(u-9)=0$

$u=1, u=9$

Since $u=3^x$ we have:

$3^x=1$ and $3^x=9$

Therefore $x = 0$ and $x = 2$.

Finding an Exponential Functions from given Conditions

To find the specific equation of a general exponential function $f(x)=k\left(b^{ax}\right)+c$ you need to find some of the four unknown parameters: *k, a , b* and *c*. Typically some specific conditions are given, such as the horizontal asymptote and a few points *(x, y)* that lie on the graph. For example, use the Horizontal Asymptote to find *c* or use the *y*-intercept (or a point where *x=0*) to find *k*.

Example 15:

The function $f(x)=k\left(b^{x}\right)$ passes through points (0, 4) and (1, 6). Write the equation of $f(x)$

Solution:

When $x = 0$ then $f(0)=4=k\times b^{0}=k$ therefore $k = 4$.

When $x = 1$ then $f(1)=6=4\times b^{1}$ then $b=6-4=2$

Therefore the equation for $f(x)=k\left(b^{x}\right)$ is $f(x)=4\times 2^{x}$

Example 16:

The function $f(x)=k\left(b^{x}\right)+c$ passes through points (0, - 8) and (1, -23) and as x is approaching infinity y is approaching -3. Write the equation of $f(x)$.

Solution:

As $x\to\infty, y\to -3$ then $y = -3$ is horizontal asymptote. Therefore $c = -3$.

When $x = 0$ then $f(0)=-8=k\times b^{0}-3\to k=-5$.

When $x = 1$ then $f(1)=23=-5\times b^{1}-3\to b=4$

Therefore the equation for $f(x)=k\left(b^{x}\right)+c$ is $f(x)=-5\left(4^{x}\right)-3$

Example 17:

The function $f(x)=k\left(b^{x}\right)+c$ passes through points (0, 11) and (-2, 35) and as x is approaching infinity y is approaching 3. Write the equation of $f(x)$.

Solution:

As $x\to\infty, y\to 3$ then $y = 3$ is horizontal asymptote. Therefore $c = 3$.

When $x = 0$ then $f(0)=11=k\times b^{0}+3\to k=8$.

When $x = -2$ then $f(-2)=35=8\times b^{-2}+3\to\ 35=\dfrac{8}{b^{2}}+3\to$

$$\to\ 32=\frac{8}{b^{2}}\to\ b^{2}=\frac{8}{32}\to\quad\to b^{2}=\frac{1}{4}\to b=\frac{1}{2}$$

Therefore the equation for $f(x)=k\left(b^{x}\right)+c$ is $f(x)=8\left(\dfrac{1}{2}\right)^{x}+3$

Example 18:

The function $f(x)=k\left(b^{ax}\right)+c$ has a horizontal asymptote at $y = 7$. The graph passes through the *y*-intercept at (0, -20), and two other points: (1, -2) and (2, 4).

If the base is known to be $b = 3$, find the exact values of k, a and c to write the equation of $f(x)$.

Solution:

horizontal asymptote at $y = 7$. Therefore $c = 7$.

When $x = 0$ then $f(0) = -20 \rightarrow k \times 3^0 + 7 = -20 \rightarrow k = -27$.

When $x = 1$ then $f(1) = -2 \rightarrow -27 \times 3^a + 7 = -2 \rightarrow -27 \times 3^a = -9 \rightarrow 3^a = \frac{1}{3} \rightarrow a = -1$

Verify when $x = 2$ then $f(2) = -27 \times 3^{-2} + 7 = -3 + 7 = 4$

Therefore the equation for $f(x) = k(b^{ax}) + c$ is $f(x) = -27(3^{-x}) + 7$

Applications of Exponential Functions

Exponential functions are one of the most important types of function used in practical applications. Exponential functions arise in many situations where the rate of change of a quantity is proportional to the amount of the quantity:

- *Exponential Growth:* a quantity exhibits exponential growth when its size after a period of time can be modeled by an exponential function whose base is greater than 1.
- *Exponential Decay:* a quantity exhibits exponential decay when its size after a period of time can be modeled by an exponential function whose base is between 0 and 1.
- *Doubling-time growth formula*: if a population of size P_o doubles every n years (or hours, or day, or any other unit of time), then the number P_n in the population at time t is given by $P(t) = P_0 \times 2^{t/n}$
- *Half-life decay formula*: if an amount of a radioactive substance of size P_o halves every h years (or hours, or day, or any other unit of time), then the amount remaining at time t is $P\left(\frac{t}{h}\right) = P_0 \times \left(\frac{1}{2}\right)^{t/h}$
- *Compound interest formula*: if an amount P_o (called the *Principal*) is invested at an annual interest rate r compounded n times a year, then in t years the investment will grow to an amount P given by $P(nt) = P_0 \times \left(1 + \frac{r}{n}\right)^{nt}$ where r is expressed as a percentage or decimal.

Example 19:

A tech startup launches a new application. The daily active user base experiences a steady exponential expansion of 20% every week. If the app initializes with 2,500 daily active users, determine the exact size of the user base after exactly 3 weeks.

Solution:

Let $N(t) = N_0 \times (1 + 20\%)^t$ the function to determine the size of the user base at time t (measured in weeks) and N_0 the initial users = 2500. Therefore after 3 weeks we have: $N(3) = 2500 \times (1 + 20\%)^3 = 4320$.

Example 20:

A mathematical cooling process follows the model $f(t) = k(b)^t + c$. The structural ambient room temperature represents a horizontal asymptote at 22°. The initial temperature at $t = 0$ is 86°. At $t = 2$ minutes, the system drops to 38°. Determine the exact time t when the temperature reaches precisely 30°.

Solution:

horizontal asymptote at $22^\circ \rightarrow c = 22^\circ$

When $t = 0$ then $f(0) = 86 \rightarrow k \times b^0 + 22 = 86 \rightarrow k = 64$

When $t = 2$ then $f(2) = 38 \rightarrow 64 \times b^2 + 22 = 38 \rightarrow 64b^2 = 16 \rightarrow b^2 = \frac{16}{64} \rightarrow b = \frac{1}{2}$

Solve for t:

$$30 = 64\left(\frac{1}{2}\right)^t + 22 \rightarrow 8 = 64\left(\frac{1}{2}\right)^t \rightarrow \frac{8}{64} = \left(\frac{1}{2}\right)^t \rightarrow \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^t \rightarrow t = 3 \text{ minutes}$$

Example 21:

A digital dataset is spreading virally over a decentralized server mesh network. The total infected nodes follow a strict doubling formula $P(t) = P_0 \times 2^{t/n}$. At hour $t = 3$, there are 1,600 infected nodes. At hour $t = 5$, there are 6,400 infected nodes. Solve for the initial node infection size P_0 and the precise doubling timeframe n.

Solution:

When $t = 3$ then $P(3) = 1600 \rightarrow P_0 \times 2^{3/n} = 1600 \rightarrow P_0 = \frac{1600}{2^{3/n}}$

When $t = 5$ then $P(5) = 6400 \rightarrow P_0 \times 2^{5/n} = 6400 \rightarrow P_0 = \frac{6400}{2^{5/n}}$

Solve for n:

$$\frac{6400}{2^{5/n}} = \frac{1600}{2^{3/n}} \rightarrow \frac{4}{2^{5/n}} = \frac{1}{2^{3/n}} \rightarrow 4\left(2^{3/n}\right) = 2^{5/n} \rightarrow 4 = 2^{2/n} \rightarrow 4^n = 4 \rightarrow n = 1$$

Solve for P_0: $P_0 = \frac{1600}{2^{3/1}} = 200$

Example 22:

An experimental chemical compound degrades in a toxic waste processing center according to the standard half-life formula $P\left(\frac{t}{h}\right) = P_0 \times \left(\frac{1}{2}\right)^{t/h}$. A technician measures the compound sample at two precise points: at hour 6, there are 72 grams remaining, and at hour 10, there are exactly 4.5 grams remaining. Find the half-life parameter h of the chemical compound.

Solution:

$$P\left(\frac{6}{h}\right) = 72 \rightarrow P_0 \times \left(\frac{1}{2}\right)^{6/h} = 72 \rightarrow P_0 = \frac{72}{\left(\frac{1}{2}\right)^{6/h}}$$

$$P\left(\frac{10}{h}\right) = 4.5 \rightarrow P_0 \times \left(\frac{1}{2}\right)^{10/h} = 4.5 \rightarrow P_0 = \frac{4.5}{\left(\frac{1}{2}\right)^{10/h}}$$

Solve for h:

$$\frac{4.5}{\left(\frac{1}{2}\right)^{10/h}} = \frac{72}{\left(\frac{1}{2}\right)^{6/h}} \rightarrow \frac{4.5}{72} = \frac{\left(\frac{1}{2}\right)^{10/h}}{\left(\frac{1}{2}\right)^{6/h}} \rightarrow \frac{1}{16} = \left(\frac{1}{2}\right)^{4/h} \rightarrow \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^{4/h} \rightarrow 4 = \frac{4}{h} \rightarrow h = 1$$

Example 23:

An investor deposits an unknown principal amount P_0 into a unique corporate bond account that compounds interest quarterly. The account accumulates wealth according to a fixed rate. After exactly 6 months, the balance is \$2,000. After exactly 12 months, the balance has expanded to \$2420.
Find the annual interest rate r of the bond.

Solution:

Using $P(nt) = P_0 \times \left(1+\frac{r}{n}\right)^{nt}$ with $n = 4,\ t = \frac{1}{2}(6\,\text{months}), t = 1(12\,\text{months})$ we have:

$$P\left(4\left(\frac{1}{2}\right)\right) = 2000 \ \rightarrow\ P_0 \times \left(1+\frac{r}{4}\right)^2 = 2000 \ \rightarrow\ P_0 = \frac{2000}{\left(1+\frac{r}{4}\right)^2}$$

$$P\left(4(1)\right) = 2420 \ \rightarrow\ P_0 \times \left(1+\frac{r}{4}\right)^4 = 2420 \ \rightarrow\ P_0 = \frac{2420}{\left(1+\frac{r}{4}\right)^4}$$

Solve for *r*:

$$\frac{2420}{\left(1+\frac{r}{4}\right)^4} = \frac{2000}{\left(1+\frac{r}{4}\right)^2} \ \rightarrow \frac{2420}{2200} = \left(1+\frac{r}{4}\right)^2 \rightarrow\ 1.21 = \left(1+\frac{r}{4}\right)^2 \ \rightarrow\ (1.1)^2 = \left(1+\frac{r}{4}\right)^2 \rightarrow$$

$$\rightarrow\ 1.1 = 1+\frac{r}{4} \ \rightarrow\ 0.1 = \frac{r}{4} \ \rightarrow r = 0.4 = 40\%$$

Miscellaneous Problems

Example 24:

Evaluate: $\dfrac{\left(3^{x+2}\right)\left(9^{x-1}\right)}{27^{\frac{x}{3}}}$

Solution:

Rewrite all terms with base 3:

$$\frac{\left(3^{x+2}\right)\left(9^{x-1}\right)}{27^{\frac{x}{3}}} = \frac{\left(3^{x+2}\right)\left(\left(3^2\right)^{x-1}\right)}{\left(3^3\right)^{\frac{x}{3}}}$$

Apply properties of exponent sans simplify: $\dfrac{\left(3^{x+2}\right)\left(9^{x-1}\right)}{27^{\frac{x}{3}}} = \dfrac{\left(3^{x+2}\right)\left(\left(3^2\right)^{x-1}\right)}{\left(3^3\right)^{\frac{x}{3}}} = \dfrac{\left(3^{x+2}\right)\left(3^{2x-2}\right)}{3^x} = \dfrac{3^{3x}}{3^x} = 3^{2x}$

Example 25:

Solve: $2^{2x+1} - 3\left(2^x\right) + 1 = 0$

Solution:

Rewrite:

$2^{2x+1} - 3\left(2^x\right) + 1 = 0$ as $2\left(2^x\right)^2 - 3\left(2^x\right) + 1 = 0$

Let $u = 2^x$ and the equation becomes:

$2u^2 - 3u + 1 = 0$

Factor:

$2u^2 - 3u + 1 = (2u - 1)(u - 1)$

So $u = \frac{1}{2}, u = 1$

Since $u = 2^x$

$2^x = \frac{1}{2}, 2^x = 1 \rightarrow x = -1,\ x = 0$

Example 26:

The function $f(x) = a(b^x) + c$ passes through points $(1,14),(2,26)$ and as $x \to \infty, y \to 2$. Write the equation of $f(x)$.

Solution:

As $x \to \infty, y \to 2$ then $y = 2$ is horizontal asymptote. Therefore $c = 2$.

When $x = 1$ then $f(1) = 14 \rightarrow ab + 2 = 14$ (equation 1)

When $x = 2$ then $f(2) = 26 \rightarrow ab^2 + 2 = 26$ (equation 2)

From equation 1 we have: $ab = 12$; substitute in equation 2 to have:

$ab^2 + 2 = 26 \rightarrow 12b + 2 = 26 \rightarrow b = 2$

Then a = 6

Therefore the equation for $f(x) = a(b^x) + c$ is $f(x) = 6(2^x) + 2$

Example 27:

The population of a city was 1,500,000 on January 1, 2020 and 3,000,000 on January 1, 2025. Assuming exponential growth of the form $P_0 \times (2)^{k \times t}$ find the year that the population will be 6,000,000.

Solution:

Let P the population in millions then $P_0 = 1.5 \times 2^{kt}$, where $t = 0$ is the time from January 1, 2020.

First we solve for k:

$3 = 1.5(2)^{5k} \rightarrow 2 = 2^{5k} \rightarrow 1 = 5k \rightarrow k = \frac{1}{5} = 0.2$

Now we solve for t:

$6 = 1.5 \times (2)^{0.2t} \rightarrow \frac{6}{1.5} = 2^{0.2t} \rightarrow 4 = 2^{0.2t} \rightarrow (2)^2 = 2^{0.2t}$.

It follows $2 = 0.2t$ therefore $t = \frac{2}{0.2} = 10$

Therefore the population will reach 6,000,000 after 10 years in 2030.

Practice Problems

Evaluate each function.

1) $k(n) = 3 \cdot 3^{n-2}$; Find $k(2)$

3

2) $h(t) = 3^{t-2}$; Find $h(-2)$ $\frac{1}{81}$

3) $g(t) = 2^t$; Find $g(1)$

2

4) $p(a) = 5^{2a}$; Find $p(0)$

1

5) $p(x) = 3^{x+2} + 1$; Find $p\left(\frac{x}{4}\right)$ $3^{\frac{8+x}{4}} + 1$

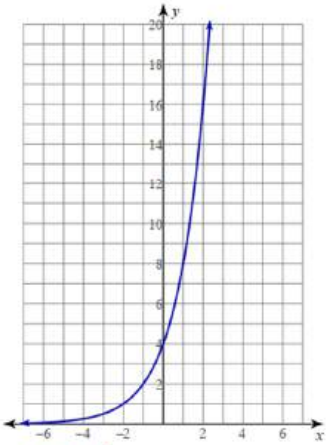
6) $h(x) = 2^{-x}$; Find $h\left(x + \frac{4}{3}\right)$ $2^{\frac{-3x-4}{3}}$

7) $h(x) = 3^x + 2$; Find $h(-x)$ $\frac{2 \cdot 3^x + 1}{3^x}$

8) $w(a) = -\frac{5}{3} \cdot 4^{a-2} + 1$; Find $w\left(a + \frac{3}{2}\right)$ $1 - \frac{5 \cdot 4^{\frac{-1+2a}{2}}}{3}$

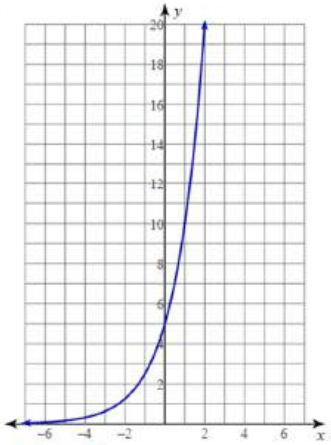
Write an equation for each graph.

9)



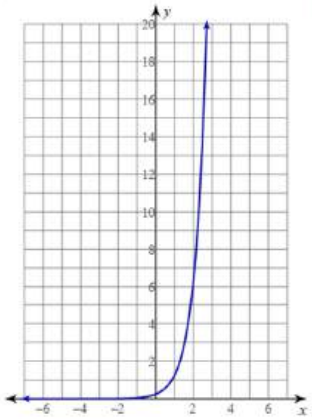
$y = 4 \cdot 2^x$

10)



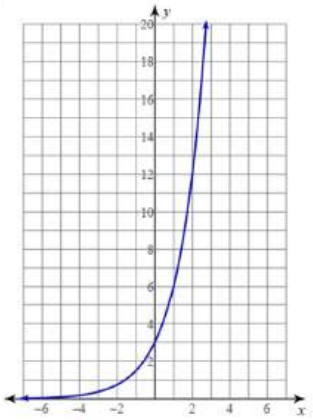
$y = 5 \cdot 2^x$

11)



$y = \frac{1}{4} \cdot 5^x$

12)



$y = 3 \cdot 2^x$

13) The exponential function $y = a \cdot 2^x + c$ has a horizontal asymptote $y = 5$ and passes through $(0, 17)$ and $(2, 53)$. Find its equation.

$y = 12 \cdot 2^x + 5$

14) The function $y = a \cdot 3^x + c$ has a horizontal asymptote $y = -4$ and passes through $(0,8)$ and $(1,32)$. Find the equation.

$y = 12 \cdot 3^x - 4$

15) For what values of a the function is a growth? $y=\left(\frac{a^2-4}{11-2a}\right)^x$

$a<-5$ or $-\frac{11}{2}<a<3$

16) For what values of a the function is a decay? $y=\left(\frac{2-a}{a+2}\right)^x$

$-2<a<0$

Solve each equation.

17) $6^{3-3x}=1$

$\{1\}$

18) $3^{-3x-1}=3^{2x+3}$ $\left\{-\frac{4}{5}\right\}$

19) $64\cdot 4^{-n}=4^{n-1}$

$\{2\}$

20) $125^{-2m+1}=\frac{1}{5}$ $\left\{\frac{2}{3}\right\}$

21) $16^{2m-2}\cdot\frac{1}{64}=1$

$\left\{\frac{7}{4}\right\}$

22) $64^{3n}\cdot 64^{-3n-1}=\left(\frac{1}{8}\right)^{n+3}$ $\{-1\}$

23) $\frac{2^3}{8^{x+3}}=16$ $\left\{-\frac{10}{3}\right\}$

24) $6^2\cdot 216^{-3v}=6^2$ $\{0\}$

25) $625^{-2m}\cdot 625^{-2m}=25^m$

$\{0\}$

26) $5^3\cdot 625^{v+2}=1$ $\left\{-\frac{11}{4}\right\}$

27) $\left(\frac{1}{81}\right)^{2p+1}\cdot 9^{-2p}=243$

$\left\{-\frac{3}{4}\right\}$

28) $9\cdot\left(\frac{1}{243}\right)^{-3a-1}=27^{-a}$ $\left\{-\frac{7}{18}\right\}$

29) Solve: $3^x=\frac{\sqrt{3}}{9}$ $-\frac{3}{2}$

30) Solve: $75\cdot 5^{3x-2}-5^{3x+1}=-50$

$x=\frac{2}{3}$

31) Solve: $6\cdot 2^{x+3}=4\cdot 7^x-2^x$

32) Solve: $\frac{8^{2-x}}{2^{2+x}}=\frac{16^{2x-1}}{4^x}$ $\frac{4}{5}$

33) Solve: $4^x+\left(2^x\right)^2-2^{2(x-2)}=124$

3

34) Solve: $4^{2x-1}-4^{2x+1}+3\cdot 2^{4x}=-\frac{3}{2}$ $\frac{1}{4}$

35) A digital dataset is spreading virally over a decentralized server mesh network. The total infected nodes follow a strict doubling formula $N(t)=N_0\cdot 2^{\frac{t}{d}}$. At hour $t=3$, there are 1,600 infected nodes. At hour $t=5$, there are 6,400 infected nodes. Solve for the initial node infection size N_0 and the precise doubling timeframe d.

$N_o=200, d=1$

36) A clinical dose of a tracking isotope contains 120 milligrams of an unstable element with a radioactive half-life of exactly 8 days. What mass of active tracking isotopes remains functional in the patient's bloodstream after a period of 24 days?

15

2.2 – Logarithmic Functions

Topic: definition, logs as inverses,properties and graphs

Review

1. Solve $27^{2x+1}=3$ $\left[\frac{2}{3}\right]$ 2. $4^{2x}-63\left(4^{x}\right)-64=0$ [3] 3. $2^{\frac{2}{3}x+1}-3\left(2^{\frac{1}{3}x}\right)-20=0$ [6]

Definition

- Let b be a positive number with $b\neq 1$. The logarithmic function with base b, denoted by $\log_b x$ is defined by $\log_b x$ if and only if $\log_b x=a\Leftrightarrow b^a=x$ .So $\log_b x$ is the exponent to which the base b must be raised to give x.

- A logarithmic function is the inverse of an exponential function.

- The argument of a logarithmic function is the input value for which the logarithm is evaluated. In the expression $\log_b\left(x+10\right)$ the argument is $\left(x+10\right)$.

- Domain: Logarithmic functions are only defined for positive values of the argument.

-Range: the range is all real numbers $\left(-\infty,\infty\right)$

Example 1:

Rewrite in logarithmic form: $3^4=81$

Solution:

Using $\log_b x=a\Leftrightarrow b^a=x$ we have: $3^4=81\Leftrightarrow\log_3 81=4$

Example 2:

Rewrite in logarithmic form: $2^x=32$

Solution:

Using $\log_b x=a\Leftrightarrow b^a=x$ we have: $2^x=32\Leftrightarrow\log_2 32=x$

Example 3:

Rewrite in logarithmic form: $a^3=b$

Solution:

Using $\log_b x=a\Leftrightarrow b^a=x$ we have: $a^3=b\Leftrightarrow\log_a b=3$

Example 4:

Rewrite in exponential form: $\log_4 16=2$

Solution:

Using $\log_b x=a\Leftrightarrow b^a=x$ we have: $\log_4 16=2\Leftrightarrow 4^2=16$

Example 5:

Rewrite in exponential form: $\log_7 x=5$

Solution: Using $\log_b x=a\Leftrightarrow b^a=x$ we have: $\log_7 x=5\Leftrightarrow 7^5=x$

Example 6:

Rewrite in exponential form: $\log_x 2 = 8$

Solution:

Using $\log_b x = a \Leftrightarrow b^a = x$ we have: $\log_x 2 = 8 \Leftrightarrow x^8 = 2$

Example 7:

Evaluate: $\log_2 x = 5$

Solution:

Using $\log_b x = a \Leftrightarrow b^a = x$ we have: $\log_2 x = 5 \Leftrightarrow 2^5 = x \rightarrow x = 32$

Example 8:

Evaluate: $\log_{13} 13\sqrt{13} = x$

Solution:

Using $\log_b x = a \Leftrightarrow b^a = x$ we have: $\log_{13} 13\sqrt{13} = x \Leftrightarrow 13^x = 13^{\frac{3}{2}} \rightarrow x = \frac{3}{2}$

Example 9:

Evaluate: $\log_x \frac{1}{4} = -2$

Solution:

Using $\log_b x = a \Leftrightarrow b^a = x$ we have: $\log_x \frac{1}{4} = -2 \Leftrightarrow x^{-2} = \frac{1}{4} \rightarrow \frac{1}{x^2} = \frac{1}{4} \rightarrow \left(\frac{1}{x}\right)^2 = \left(\frac{1}{2}\right)^2 \rightarrow x = 2$

Example 10:

Find the domain $y = \log_5 (x+2) - 3$

Solution:

Domain: $x + 2 > 0 \rightarrow x > -2$

Example 11:

Find the domain $y = \ln\left(x^2 - 4\right) - 1$

Solution:

$x^2 - 4 > 0$. It is a parabola open upward with roots $x = \pm 2$. Therefore domain: $x < -2$ or $x > 2$

Properties of Logarithms-Common and Natural Logarithms

Common logarithm: the logarithm with base 10 is called the common logarithm and is denoted by omitting the base: $\log_{10} x = \log x$

Natural logarithm: the logarithm with base e is called the natural logarithm and is denoted by $\log_e x = \ln x$

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x$

Example 12:

Find the domain $y=\dfrac{\sqrt{x^2-9}}{\log_2\left(x^2-4\right)}$

Solution:

Numerator: $x^2-9\geq 0$. It is a parabola open upward with roots $x=\pm 3$. Therefore $x\leq -3$ or $x\geq 3$

Denominator: $x^2-4>0$ and $\log_2\left(x^2-4\right)\neq 0$. Therefore $x<-2$ or $x>2$ and $x^2-4\neq 1\rightarrow x\neq\pm\sqrt{5}$

(property 1)

Domain: inequality that satisfy all conditions is: $x\leq -3$ or $x\geq 3$

Example 13:

Solve: $\log_9 x=1$

Solution:

By property 2: $x=9^1=9$

Example 14:

Solve: $\log_{16}\sqrt{2}=x$

Solution:

By using definition:

$$16^x=\sqrt{2}\ \rightarrow\ \left(2^4\right)^x=2^{\frac{1}{2}}\ \rightarrow 2^{4x}=2^{\frac{1}{2}}\ \rightarrow 4x=\frac{1}{2}\ \rightarrow x=\frac{1}{8}$$

Example 15:

Solve: $\log_x 81=4$

Solution:

By using definition:

$$x^4=81\ \rightarrow\ x^4=3^4\ \rightarrow x=3$$

Example 16:

Prove property 3: $\log_b b^x=x$

Solution:

Let $y=\log_b b^x$.

Rewrite in equivalent exponential form: $y=\log_b b^x\ \Leftrightarrow\ b^y=b^x$

If the bases are identical then their exponents must be equal: $y=x$

Substitute $y=x$ into $y=\log_b b^x$ to have $\log_b b^x=x$.

Example 17:

Prove property 4: $b^{\log_b x}=x$

Solution:

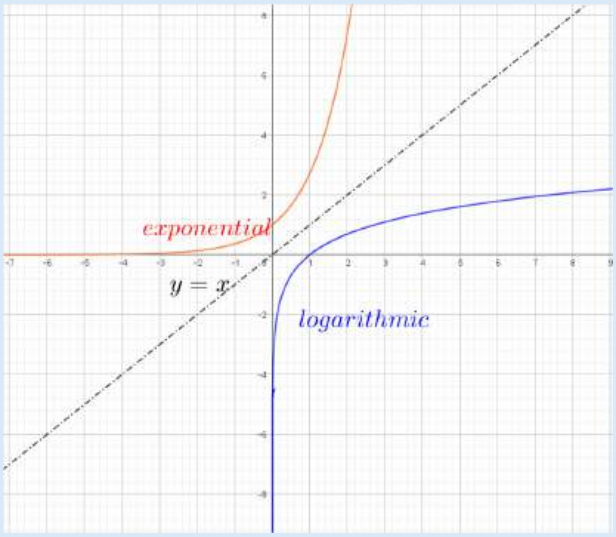
Let $y=\log_b x$.

Rewrite in equivalent exponential form: $y=\log_b x\ \Leftrightarrow\ b^y=x$

Substitute $y=\log_b x$ into $b^y=x$ to have $b^{\log_b x}=x$.

Graphs of Logarithmic Functions

Since the logarithmic function is the inverse of the exponential function the graphs are symmetrical to the line $y = x$.

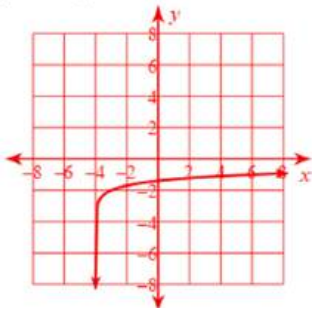


Example 18:

Graph $\log(x+4)-2$

Solution:

Shift to the left 4 units then 2 units down

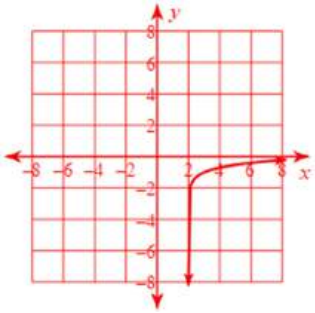


Example 19:

Graph $\log(x-2)-1$

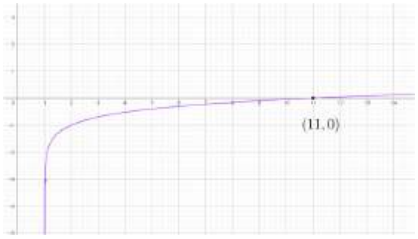
Solution:

Shift to the right 2 units then 1 unit down



Example 20:

Given the graph of $\log(x-a)-b$, find *a* and *b*.



Solution:

The base is 10; *x*-intercept (11,0) then graph shift 1 unit to the right and 1 unit down so:

$\log(x-a)-b=\log(x-1)-1$. Therefore *a = 1* and *b = 1*.

Example 21:

A logarithmic function is defined by the equation $\log_3(x-a)+b$. The graph of the function has a vertical asymptote at *x = 2* and passes through the point (5, 3). Find the exact values of find *a* and *b*..

Solution:

vertical asymptote at *x = 2* implies $x-2=0$ then it follows *a = 2*.

Use (5, 3) to find *b*:

$\log_3(5-2)+b=3 \quad \rightarrow \quad 1+b=3 \quad \rightarrow \quad b=2$

Example 22:

The function $y=\log_2(x+a)+b$ passes through the points (1, 3 and (5, 4). Find the exact values of *a* and *b*..

Solution:

Using the points (1, 3 and (5, 4):

$$\begin{matrix}\log_2(1+a)+b=3\\ \log_2(5+a)+b=4\end{matrix} \rightarrow \begin{matrix}\log_2(1+a)=3-b\\ \log_2(5+a)=4-b\end{matrix} \rightarrow \begin{matrix}(1+a)=2^{(3-b)}\\ (5+a)=2^{(4-b)}\end{matrix} \rightarrow \begin{matrix}(1+a)=\frac{2^3}{2^b}\\ (5+a)=\frac{2^4}{2^b}\end{matrix} \rightarrow \begin{matrix}(1+a)=\frac{8}{2^b}\\ (5+a)=\frac{16}{2^b}\end{matrix}$$

Subtract $(1+a)=\frac{8}{2^b}$ from $(5+a)=\frac{16}{2^b}$ to have $4=\frac{8}{2^b} \rightarrow 1=\frac{2}{2^b} \rightarrow b=1$.

Solve for *a*: $(1+a)=\frac{8}{2} \rightarrow 1+a=4 \rightarrow a=3$

Example 23:

Find the inverse $y=2^x-5$

Solution:

Rewrite $y=2^x-5$ as $x=2^y-5$

Solve for *y*:

$x+5=2^y$ take $\log_2$ on both sides

$\log_2(x+5)=\log_2(2^y)$

$\log_2(x+5)=y\log_2 2$ then the inverse is $y=\log_2(x+5)$

Example 24:

Find the inverse $y = 8\log_6(x+9)$

Solution:

Rewrite $y = 8\log_6(x+9)$ as $x = 8\log_6(y+9)$

Solve for *y*:

$\frac{x}{8} = \log_6(y+9)$ using definition

$6^{\frac{x}{8}} = y+9$

$6^{\frac{x}{8}} - 9 = y$ then the inverse is $y = 6^{\frac{x}{8}} - 9$

Example 25:

Find the inverse $y = \left(\frac{e^x - 6}{-4}\right)^{\frac{1}{5}}$

Solution:

Rewrite $y = \left(\frac{e^x - 6}{-4}\right)^{\frac{1}{5}}$ as $x = \left(\frac{e^y - 6}{-4}\right)^{\frac{1}{5}}$

Solve for *y*:

$(x)^5 = \left[\left(\frac{e^y - 6}{-4}\right)^{\frac{1}{5}}\right]^5$ raise to the 5th power

$x^5 = \frac{e^y - 6}{-4}$

$-4x^5 = e^y - 6$

$6 - 4x^5 = e^y$ take ln on both side

$\ln\left(6 - 4x^5\right) = \ln\left(e^y\right)$ then the inverse is $y = \ln\left(6 - 4x^5\right)$

Miscellaneous Problems

Example 26:

Find the domain $y = \ln\sqrt{x^2 - 5x + 6}$

Solution:

The argument $x^2 - 5x + 6 > 0$.

Factor: $(x-2)(x-3) > 0$

It is a parabola open upward with roots $x = 2, x = 3$. Therefore $x < 2$ or $x > 3$

Example 27:

Find the domain $y = \log_5\left(\frac{x+2}{x^2-16}\right)$

Solution:

The argument $\frac{x+2}{x^2-16} > 0$.

Factor: $\frac{x+2}{(x-4)(x+4)} > 0$

Sign chart:

Interval	$x+2$	$x-4$	$x+4$	overall
$(\infty,-4)$	-	-	-	-
$(-4,-2)$	-	-	+	+
$(-2,4)$	+	-	+	-
$(4,\infty)$	+	+	+	+

Therefore $-4 < x < -2$ or $x > 4$

Example 28:

Solve $\log_2(x+1) = \log_4 16$

Solution:

$\log_4 16 = \log_4 4^2 = 2$ (by property 3). then

$\log_2(x+1) = 2$ using definition

$x+1 = 2^2 \rightarrow x = 3$

Example 29:

Solve $\log_3 x = \log_9 81$

Solution:

$\log_9 81 = \log_9 9^2 = 2$, then

$\log_3 x = 2$ using definition

$x = 3^2 \rightarrow x = 9$

Example 30:

Solve $\log_3 x = \log_x 27$

Solution:

Let $\log_3 x = y$, then $x = 3^y$ substitute to have:

$\log_x 27 = \log_{3^y} 27 = y$; using definition we have:

$(3^y)^y = 27 \rightarrow 3^{y^2} = 3^3 \rightarrow y^2 = 3 \rightarrow y = \pm\sqrt{3}$

Since $x = 3^y \rightarrow x = 3^{\sqrt{3}}$ or $x = 3^{-\sqrt{3}}$

Example 31:

Find the logarithmic equation $y = a\log_2(x-h)$ whose vertical asymptote is $x=3$ and which passes through point (11,4).

Solution:

Asymptote $x=3$ implies $h = 3$, so $y = a\log_2(x-3)$.

Using point (11, 4) we have: $4 = a\log_2(11-3) \rightarrow 4 = a\log_2 2^3 \rightarrow 4 = 3a \rightarrow a = \frac{4}{3}$

Therefore $y = \frac{4}{3}\log_2(x-3)$

Example 32:

Given $y = a\log_b(x)$ which passes through points $(b,5)$, (b^2,c) and $(32,25)$, find *a, b* and *c*.

Solution:

At $x=b$ we have $5 = a\log_b(b) \rightarrow a = 5$.

At $x=10$ we have $c = 5\log_b(b^2) \rightarrow c = 5(2)\log_b(b) \rightarrow c = 10$.

At $x=32$ we have $25 = 5\log_b(32) \rightarrow 5 = \log_b(2^5) \rightarrow 5 = 5\log_b 2 \rightarrow b = 2$.

Example 33:

Find the inverse of $y = \log_2(-2x-3)$

Solution:

Rewrite $y = \log_2(-2x-3)$ as $x = \log_2(-2y-3)$

Solve for *y*:

using definition

$2^x = -2y-3$

$2^x + 3 = -2y$ then the inverse is $y = -\frac{2^x+3}{2}$

Example 34:

Find the inverse of $y = \log_5\left(\frac{e^x+9}{3}\right)$

Solution:

Rewrite $y = \log_5\left(\frac{e^x+9}{3}\right)$ as $x = \log_5\left(\frac{e^y+9}{3}\right)$

Solve for *y*:

using definition

$5^x = \frac{e^y+9}{3}$

$3(5^x) = e^y + 9$

$3(5^x) - 9 = e^y$

Take ln on both sides: $\ln\left[3(5^x)-9\right] = \ln(e^y) \rightarrow$ the inverse is $y = \ln\left[3(5^x)-9\right]$

Practice Problems

1. Find the domain of $f(x)=\log_2\left(\frac{x^2-9}{x-4}\right)$ $\left[(-3,3)\cup(4,\infty)\right]$
2. Find the domain of $f(x)=\log_3\left(\sqrt{\frac{x-1}{x+3}}\right)$ $\left[(-\infty,-3)\cup(1,\infty)\right]$
3. Find the domain of $f(x)=\ln\left(\frac{\sqrt{x-2}}{\sqrt{x^2-9}}\right)$ $\left[(3,\infty)\right]$
4. Find the domain of $f(x)=\log_7\left(\frac{x^2-4x+3}{x^2-9}\right)$ $\left[(-\infty,-3)\cup(1,3)\cup(3,\infty)\right]$
5. Find the domain of $f(x)=\ln\left[\ln\left(x^2-4x+5\right)\right]$ $\left[x\neq 2\right]$
6. Find the domain of $f(x)=\log_2\left(\frac{\sqrt{x^2-1}}{x-2}\right)$ $\left[(2,\infty)\right]$
7. Solve $\log_2 x=\log_x 16$ $[4]$
8. Solve $\log_5 x=\log_x 125$ $\left[5\sqrt{5}\text{ or }5^{\frac{3}{2}}\right]$
9. Solve $\log_4 x=\log_x 64$ $\left[4^{\sqrt{3}}\text{ or }4^{-\sqrt{3}}\right]$
10. Solve $\log_2(x+1)=\log_{x+1}8$ $\left[2^{\sqrt{3}}-1\text{ or }2^{-\sqrt{3}}-1\right]$
11. Solve $\log_2 64=\log_4 x+1$ $\left[4^{\frac{-1+\sqrt{13}}{2}}\text{ or }4^{\frac{-1-\sqrt{13}}{2}}\right]$
12. Solve $\log_3(2x)=\log_{2x}27$ $\left[\frac{3^{\sqrt{3}}}{2}\text{ or }\frac{3^{-\sqrt{3}}}{2}\right]$
13. Find the logarithmic equation $y=a\log_3(x-h)$ whose vertical asymptote is $x=-2$ and which passes through (7,6). $\left[y=2\log_3(x+2)\right]$
14. Find the logarithmic equation $y=a\log_5(x-h)+k$ whose vertical asymptote is $x=4$, passes through (9, 1), and has a y-intercept of -1. $\left[y=\log_5(x-4)-1\right]$
15. Find the logarithmic equation $y=a\log_2(x-h)+b$ whose vertical asymptote is $x=-5$, passes through (3,4) and passes through (11, 7). $\left[y=\log_2(x+5)+1\right]$
16. Find the logarithmic equation $y=a\log_4(x-h)$ whose graph has a vertical asymptote at $x=0$, passes through (16, -6) and is reflected across the x-axis. $\left[y=-3\log_4 x\right]$
17. Find the logarithmic equation $y=a\log(x-h)+b$ whose vertical asymptote is $x=-1$, passes through (9, 5), and whose graph is horizontally compressed by a factor of $\frac{1}{2}$ from the parent function $\log x$. $\left[y=2\log(x+1)+3\right]$

18. Find the logarithmic equation $y = a\log_2(x-h)$ that has a vertical asymptote at $x = -3$, passes through (5,8) and also passes through (13, 12) $\left[y = 4\log_2(x+3)\right]$

19. Find the logarithmic equation $y = a\log_3(x-h)+b$ $y = a\log_3(x-h)+b$ whose vertical asymptote is $x = 2$, passes through (5, 0) and has an x-intercept at $x = 5$. $\left[y = \log_3(x-2)-1\right]$

20. Find the logarithmic equation $y = a\log_2(x-h)+b$ whose vertical asymptote is $x = 1$, passes through (3, -2) and has a y-intercept of 2. $\left[y = -2\log_2(x-1)+2\right]$

Find the inverse of each function.

21) $y = 2^x - 5$

$y = \log_2(x+5)$

22) $y = 5^x + 8$

$y = \log_5(x-8)$

23) $y = 2^x + 5$

$y = \log_2(x-5)$

24) $y = 3^{\frac{x}{2}}$

$y = \log_3 x^2$

25) $y = 8\log_6(x+9)$

$y = 6^{\frac{x}{8}} - 9$

26) $y = \log_{\frac{1}{5}}(x^2+9)$ $y = \left(\left(\frac{1}{5}\right)^x - 9\right)^{\frac{1}{2}}$

27) $y = \log_2(-2x-3)$

$y = \frac{2^x+3}{-2}$

28) $y = \log(x^5+5)$

$y = (10^x - 5)^{\frac{1}{5}}$

29) $y = \left(\frac{e^x-6}{-4}\right)^{\frac{1}{5}}$

$y = \ln(-4x^5+6)$

30) $y = \left(\frac{e^x-3}{-3}\right)^{\frac{1}{3}}$

$y = \ln(-3x^3+3)$

31) $y = \left(\frac{e^x+2}{-2}\right)^{\frac{1}{4}}$

$y = \ln(-2x^4-2)$

32) $y = \log_5 \frac{e^x+9}{3}$

$y = \ln(3\cdot 5^x - 9)$

2.3-Laws of Logarithms

Topic: expand, condense, change of base, applications

Review

1. Solve for x: $\log_{169} 13 = x$ $\left[\frac{1}{2}\right]$

2. Solve for x: $\log_x 64 = 6$ [2]

3. Evaluate: $\log_2 16 + \log_{16} 2$ $\left[\frac{17}{4}\right]$

4. If $f(x) = \log_9 x$, find $f^{-1}(2) + f^{-1}\left(\frac{1}{2}\right)$ [84]

Law of Logarithms

Let b be a positive number with $b > 0$ and $b \neq 1$. Let A,B and C be any real numbers with $A>0$ and $B>0$ then:

1. $\log_b (AB) = \log_b A + \log_b B$
2. $\log_b \left(\frac{A}{B}\right) = \log_b A - \log_b B$
3. $\log_b (A)^C = C \log_b A$

Example 1:

prove $\log_b (AB) = \log_b A + \log_b B$.

Solution:

Let $\log_b A = x$ and $\log_b B = y$ then

$b^x = A$ and $b^y = B$

$AB = b^x b^y = b^{x+y}$

$\log_b AB = \log_b b^{x+y}$

$\log_b AB = x + y$

$\log_b AB = \log_b A + \log_b B$

Example 2:

Expand $\log_3 a^2 b^3 \sqrt{c}$

Solution:

$$\log_3 a^2 b^3 \sqrt{c} = \log_3 a^2 + \log_3 b^3 + \log_3 \sqrt{c} =$$

$$= 2\log_3 a + 3\log_3 b + \frac{1}{2}\log_3 c$$

Example 3:

Expand $\log \frac{\sqrt{a^2 b^3}}{x^2 y}$

Solution:

$$\log \frac{\sqrt{a^2 b^3}}{x^2 y} = \log \sqrt{a^2 b^3} - \log x^2 y = \frac{1}{2}(\log a^2 + 3\log b) - (2\log x + \log y) = \frac{1}{2}(2\log a) + \frac{1}{2}(3\log b) - 2\log x - \log y =$$

$$= \log a + \frac{3}{2}\log b - 2\log x - \log y$$

Example 4:

Condense $6\ln x+3\ln 5-2\ln 8$

Solution:

$$6\ln x+3\ln 5-2\ln 8=\ln x^6+\ln 5^3-\ln 8^2=\ln\left(x^6\right)\left(5^3\right)-\ln 8^2=\ln\left[\frac{\left(x^6\right)\left(5^3\right)}{8^2}\right]$$

Example 5:

Condense $\ln a-3\ln b-\frac{2}{3}\ln c$

Solution:

$$\ln a-3\ln b-\frac{2}{3}\ln c=\ln a-\left(3\ln b+\frac{2}{3}\ln c\right)=\ln a-\left(\ln b^3+\ln c^{\frac{2}{3}}\right)=\ln\left(\frac{a}{b^3c^{\frac{2}{3}}}\right)$$

Example 6:

Given $\log_4 7\approx 1.4;\ \log_4 9\approx 1.6;\ \log_4 6\approx 1.3$. Find $\log_4\frac{1}{196}$

Solution:

$$\log_4\frac{1}{196}=\log_4 196^{-1}=\log_4(49\times 4)=-\left(\log_4 49+\log_4 4\right)=-\left(\log_4 7^2+\log_4 4\right)=-\left(2\log_4 7+1\right)=-\left(2(1.4)+1\right)=-3.8$$

Example 7:

Given $\log_3 4\approx 1.3;\ \log_3 10\approx 2.1;\ \log_3 11\approx 2.2$. Find $\log_3\frac{9}{110}$

Solution:

$$\log_3\frac{9}{110}=\log_3 9-\log_3 110=\log_3 3^2-\log_3(11\times 10)=2\log_3 3-\left(\log_3 11+\log_3 10\right)=2-(2.2+2.1)=-2.3$$

Example 8:

Given $\log_9 11\approx 1.1;\log_9 6\approx 0.8;\log_9 8\approx 0.9$. Find $\log_9\frac{1}{704}$

Solution:

$$\log_9\frac{1}{704}=\log_9 704^{-1}=-\log_9(11\times 64)=-\left(\log_9 11+\log_9 64\right)=-\left(1.1+2\log_9 8\right)=-(1.1+1.8)=-2.9$$

Example 9:

Given $\log_9 10=A;\log_9 12=P;\log_9 11=Q$. Find $\log_9 17280$ in terms of the letter.

Solution:

$$\log_9 17280=\log_9\left(12^3\times 10\right)=\log_9 12^3+\log_9 10=3\log_9 12+M=3P+M$$

Example 10:

Given $\log_9 6=A;\log_9 5=P;\log_9 8=Q$. Find $\log_9\frac{15}{32}$ in terms of the letter.

Solution:

$$\log_9\frac{15}{32}=\log_9\frac{15\times 2}{32\times 2}=\log_9\frac{30}{64}=\log_9 30-\log_9 64=\log_9(5\times 6)-\log_9 8^2=\log_9 5+\log_9 6-2\log_9 8=P+A-2Q$$

Change of Base Formula

The change of base formula lets you rewrite a logarithm in one base using another base (usually one your calculator can handle):

$\log_b x = \dfrac{\log_a x}{\log_a b}$ where $b =$ original base; $a =$ new base

Example 11:

Show that $\log e = \dfrac{1}{\ln 10}$.

Solution:

Using $\log_b x = \dfrac{\log_a x}{\log_a b}$ then $\log e = \dfrac{\ln e}{\ln 10} = \dfrac{1}{\ln 10}$

Example 12:

Evaluate $\dfrac{\log_2 9}{\log_2 3} + \dfrac{\log_5 25}{\log_5 5}$.

Solution:

$$\frac{\log_2 9}{\log_2 3} + \frac{\log_5 25}{\log_5 5} = \frac{\log_2 3^2}{\log_2 3} + \frac{\log_5 5^2}{\log_5 5} = \frac{2\log_2 3}{\log_2 3} + \frac{2\log_5 5}{\log_5 5} = 2 + 2 = 4$$

Example 13:

Find base b such that $\log_b 16 = \dfrac{4}{3}$.

Solution:

$$\log_b 16 = \frac{4}{3} \rightarrow \frac{\log_2 16}{\log_2 b} = \frac{4}{3} \rightarrow \frac{\log_2 2^4}{\log_2 b} = \frac{4}{3} \rightarrow \frac{4}{\log_2 b} = \frac{4}{3} \rightarrow \log_2 b = 3 \rightarrow b = 8$$

Example 14:

Simplify $(\log_2 3)(\log_3 5)(\log_5 8)$.

Solution:

$$(\log_2 3)(\log_3 5)(\log_5 8) = (\log_2 3)\left(\frac{\log_2 5}{\log_2 3}\right)\left(\frac{\log_2 8}{\log_2 5}\right) = \log_2 8 = 3$$

Example 15:

Prove that for any positive numbers $a, b, c \neq 1$: $(\log_a b)(\log_b c)(\log_c a) = 1$.

Solution:

$$(\log_a b)(\log_b c)(\log_c a) = \left(\frac{\log_a b}{1}\right)\left(\frac{\log_a c}{\log_a b}\right)\left(\frac{\log_a a}{\log_a c}\right) = 1$$

Example 16:

Evaluate: $(\log_2 3)(\log_3 4)(\log_4 5)...(\log_{99} 100)$.

Solution:

Using base ln:

$$\left(\frac{\ln 3}{\ln 2}\right)\left(\frac{\ln 4}{\ln 3}\right)\left(\frac{\ln 5}{\ln 4}\right)...\left(\frac{\ln 99}{\ln 98}\right)\left(\frac{\ln 100}{\ln 99}\right)=\left(\frac{\ln 100}{\ln 2}\right)$$

Taking $\log_2$

$$\left(\frac{\ln 100}{\ln 2}\right)=\log_2 100$$

Example 17:

Prove that $(\log_2 3)(\log_3 5)+(\log_2 5)(\log_5 3)=\log_2 15$.

Solution:

Using base ln:

$$(\log_2 3)(\log_3 5)+(\log_2 5)(\log_5 3)=$$

$$=(\log_2 3)\left(\frac{\log_2 5}{\log_2 3}\right)+\left(\frac{\log_2 5}{\log_2 2}\right)\left(\frac{\log_2 3}{\log_2 5}\right)=$$

$$=\left(\cancel{\log_2 3}\right)\left(\frac{\log_2 5}{\cancel{\log_2 3}}\right)+\left(\frac{\cancel{\log_2 5}}{\log_2 2}\right)\left(\frac{\log_2 3}{\cancel{\log_2 5}}\right)=$$

$$=\log_2 5+\log_2 3=\log_2 15$$

Example 18:

Find all real $x > 0$ such that $(\log_2 x)(\log_4 x)(\log_8 x)=1$.

Solution:

$$(\log_2 x)(\log_4 x)(\log_8 x)=1$$

$$(\log_2 x)\left(\frac{\log_2 x}{\log_2 4}\right)\left(\frac{\log_2 x}{\log_2 8}\right)=1$$

$$(\log_2 x)\left(\frac{\log_2 x}{2\log_2 2}\right)\left(\frac{\log_2 x}{3\log_2 2}\right)=1$$

$$\frac{(\log_2 x)^3}{6}=1$$

$$(\log_2 x)^3=6$$

$$(\log_2 x)=\sqrt[3]{6}$$

$$x=2^{\sqrt[3]{6}}$$

Practice Problems

Condense each expression to a single logarithm.

1) $\frac{3\log_6 7}{2} + \frac{\log_6 10}{2} + \frac{\log_6 3}{2}$

$\log_6\left(7\sqrt{210}\right)$

2) $18\log_2 c + 18\log_2 a - 3\log_2 b$

$\log_2 \frac{c^{18}a^{18}}{b^3}$

3) $24\log_9 8 - 6\log_9 7 - 6\log_9 5$

$\log_9 \frac{8^{24}}{5^6 \cdot 7^6}$

4) $\frac{\log_4 10}{3} + \frac{\log_4 7}{3} + \frac{\log_4 3}{3} + \frac{\log_4 11}{3}$

$\log_4 \sqrt[3]{2310}$

5) $3\log_5 w + 9\log_5 u + 3\log_5 v$

$\log_5\left(w^3 v^3 u^9\right)$

6) $5\log_8 u + 5\log_8 w - 30\log_8 v$

$\log_8 \frac{w^5 u^5}{v^{30}}$

Expand each logarithm.

7) $\log_2\left(\frac{a \cdot c}{b^5}\right)^5$

$5\log_2 a + 5\log_2 c - 25\log_2 b$

8) $\log \sqrt{5 \cdot 12 \cdot 7 \cdot 6}$

$\frac{\log 5}{2} + \frac{\log 12}{2} + \frac{\log 7}{2} + \frac{\log 6}{2}$

9) $\log_6\left(3 \cdot 7^3 \cdot 10\right)^3$

$3\log_6 3 + 9\log_6 7 + 3\log_6 10$

10) $\log_3\left(6 \cdot 11 \cdot 5^3\right)^3$

$3\log_3 6 + 3\log_3 11 + 9\log_3 5$

11) $\log_7\left(3 \cdot 2^6 \cdot 5\right)^3$

$3\log_7 3 + 18\log_7 2 + 3\log_7 5$

12) $\log\left(a \cdot b \cdot d \cdot c^2\right)$

$\log a + \log b + \log d + 2\log c$

Use the properties of logarithms and the values below to find the logarithm indicated. Do not use a calculator to evaluate the logs.

13) $\log_3 4 \approx 1.3$
$\log_3 10 \approx 2.1$
$\log_3 7 \approx 1.8$
Find $\log_3 \frac{4}{27}$

-1.7

14) $\log_6 8 \approx 1.2$
$\log_6 10 \approx 1.3$
$\log_6 7 \approx 1.1$
Find $\log_6 \frac{50}{21}$

0.5

15) $\log_7 4 \approx 0.7$
$\log_7 6 \approx 0.9$
$\log_7 9 \approx 1.1$
Find $\log_7 \frac{1}{486}$

-3.1

16) $\log_3 10 \approx 2.1$
$\log_3 4 \approx 1.3$
$\log_3 11 \approx 2.2$
Find $\log_3 \frac{4}{99}$

-2.9

17) $\log_3 8 \approx 1.9$
$\log_3 10 \approx 2.1$
$\log_3 7 \approx 1.8$
Find $\log_3 \frac{196}{5}$

3.4

18) $\log_3 8 \approx 1.9$
$\log_3 7 \approx 1.8$
$\log_3 10 \approx 2.1$
Find $\log_3 \frac{9}{100}$

-2.2

Use the properties of logarithms and the logarithms provided to rewrite each logarithm in terms of the variables given.

19) $\log_4 9 = R$
$\log_4 10 = S$
$\log_4 6 = T$
Find $\log_4 \frac{1}{240}$

$-1 - S - T$

20) $\log_7 8 = U$
$\log_7 6 = V$
$\log_7 9 = W$
Find $\log_7 \frac{1}{252}$

$-1 - 2V$

21) $\log_9 4 = X$
$\log_9 6 = Y$
$\log_9 11 = Z$
Find $\log_9 \frac{6}{1331}$

$Y - 3Z$

22) $\log_9 8 = U$
$\log_9 11 = V$
$\log_9 10 = W$
Find $\log_9 \frac{16}{25}$

$2U - 2W$

23) $\log_7 11 = A$
$\log_7 9 = B$
$\log_7 6 = C$
Find $\log_7 \frac{49}{36}$

$2 - 2C$

24) $\log_6 4 = A$
$\log_6 7 = B$
$\log_6 10 = C$
Find $\log_6 \frac{98}{5}$

$A + 2B - C$

25) Show that:
$\log(x^2 - 1) - \log(x + 1) = \log(x - 1)$

26) Show that
$-\ln\left(x - \sqrt{x^2 - 1}\right) = \ln\left(x + \sqrt{x^2 - 1}\right)$

27) Evaluate $\frac{1}{\log_2 3}+\frac{1}{\log_3 4}+\frac{1}{\log_4 5}$

$\frac{\ln 2}{\ln 3}+\frac{\ln 3}{\ln 4}+\frac{\ln 4}{\ln 5}$

28) Evaluate: $(\log_2 3)(\log_3 7)(\log_7 16)$

4

29) Solve: $(\log_2 x)+(\log_4 x)+(\log_8 x)=7$

$x=2^{\frac{42}{11}}$

30) Solve:
$(\log_3 x)+(\log_9 x)+(\log_{27} x)=6$

$x=3^{\frac{36}{11}}$

2.4 – Exponential and Logarithmic Equations

Topic: solving equations, applications

Review

1) $\log_3 11 \approx 2.2; \log_3 4 \approx 1.3; \log_3 10 \approx 2.1$

Find $\log_3 \frac{605}{2}$

5.2

2) $\log_5 11 \approx 1.5; \log_5 6 \approx 1.1; \log_5 8 \approx 1.3$

Find $\log_5 \frac{180}{11}$

1.7

3) $\log_3 10 = A; \log_3 4 = B; \log_3 11 = C$

Find $\log_3 \frac{33}{16}$

$C + 1 - 2B$

4) $\log_4 11 = X; \log_4 9 = Y; \log_4 6 = Z$

Find $\log_4 \frac{121}{16}$

$2X - 2$

5) Evaluate: $\log_{x+3}(x^2 + 6x + 9)$

2

6) Evaluate: $\log_{a+1}(a^3 + 3a^2 + 3a + 1)$

3

Find the inverse:

7) $y = \log_3(x^5 - 3) - 1$ $y = (3^{x+1} + 3)^{\frac{1}{5}}$

8) $y = (3^x + 8)^{\frac{1}{2}}$ $y = \log_3(x^2 - 8)$

Solve:

9) $36 \cdot \left(\frac{1}{216}\right)^{-2x} = 1$ $\left\{-\frac{1}{3}\right\}$

10) $243 \cdot \left(\frac{1}{27}\right)^{-3n-2} = 81^{3n}$ $\left\{\frac{11}{3}\right\}$

Solving Exponential Equations Requiring Logarithms

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

Example 1:

Solve $3^{2x} = 7$

Solution:

$\log_3$ on both sides: $\log_3(3^{2x}) = \log_3 7 \rightarrow 2x = \log_3 7 \rightarrow x = \frac{\log_3 7}{2}$

Example 2:

Solve $9^{n-10} - 2 = 92$

Solution:

$9^{n-10} - 2 = 92 \rightarrow 9^{n-10} = 94$. $\log_9$ on both sides: $\log_9(9^{n-10}) = \log_9 94 \rightarrow n - 10 = \log_9 94 \rightarrow n = \log_9 94 + 10$

Example 3:

Solve $3^x + \frac{12}{3^x} = 8$

Solution:

Multiply both sides by 3^x: $3^x\left(3^x+\frac{12}{3^x}\right)=(8)3^x$. We have:

$\left(3^x\right)^2+12=(8)3^x$; rearrange

$\left(3^x\right)^2-8\left(3^x\right)+12=0$; let $u=3^x$ then

$u^2-8u+12=0$ factor

$(u-6)(u-2)=0$ then $u=6, u=2$

Since $u=3^x$ we have

$3^x=6$ and $3^x=2$ then $\log_3$ both sides

$x=\log_3 6$ and $x=\log_3 2$

Example 4:

Solve $6^x+9^x=2^{2x+1}$

Solution:

$6^x+9^x=2^{2x+1}$

$6^x+9^x=2\left(2^2\right)^x$

$6^x+9^x=2\left(4^x\right)$

$\frac{6^x}{4^x}+\frac{9^x}{4^x}=2$

$\left(\frac{3}{2}\right)^x+\left(\frac{3}{2}\right)^{2x}=2$

$\left(\frac{3}{2}\right)^{2x}+\left(\frac{3}{2}\right)^x-2=0$ let $u=\left(\frac{3}{2}\right)^x$

$u^2+u-2=0$

$(u+2)(u-1)=0$

$u=-2, u=1$

Since $u=\left(\frac{3}{2}\right)^x$ we have

$\left(\frac{3}{2}\right)^x=-2 \rightarrow$ no solution , $\left(\frac{3}{2}\right)^x=1 \rightarrow x=0$

Solving Logarithmic Equations

Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.

Write the equation in exponential form (or raise the base to each side of the equation).

Solve for the variable.

Example 5:

Solve: $\log_x(2x-3)=1$

Solution:

Using definition:

$(2x-3)=x$

$3x=3$

$x=1$

Example 6:

Solve: $\log\left(x^2+8x\right)=\log(-x)$

Solution:

Same base (10) implies same arguments:

$x^2+8x=-x$

$x^2+9x=0$

$x(x+9)=0$

$\cancel{x=0}$ not a valid solution; $x=-9$

Example 7:

Solve: $\ln(7x+1)=\ln\left(x^2-17\right)$

Solution:

Same base (e) implies same arguments:

$7x+1=x^2-17$

$x^2-7x-18=0$

$(x+2)(x-9)=0$

$\cancel{x=-2}$ not a valid solution; $x=9$

Example 8:

Solve: $-6\log_6(-4x-4)+4=-2$

Solution:

$-6\log_6(-4x-4)+4=-2$

$-6\log_6(-4x-4)=-2-4$

$\log_6(-4x-4)=1$ (apply definition)

$-4x-4=6$

$-4x=10$

$x=-\frac{5}{2}$

Example 9:

Solve: $\ln 10-\ln(2-3x)=\ln 9$

Solution:

$\ln 10-\ln(2-3x)=\ln 9 \quad \rightarrow \quad \ln\left(\frac{10}{2-3x}\right)=\ln 9 \quad \frac{10}{2-3x}=9 \quad \rightarrow 10=18-27x$

$x=\frac{8}{27}$

Applications

Here's a brief overview of the real-world applications:

Psycology: The *Law of Forgetting* describes how a person's retained performance on a learned task declines over time. Specifically, if the initial performance level is P_0, then after t months, the new performance level P can be predicted using the logarithmic relationship: $\log P = \log P_0 - c\log(t+1)$

Here, c is a task-specific constant that determines the rate of forgetting, larger values of c indicate faster memory decay. The formula implies that forgetting is rapid initially and then slows down as time increases.

Example 10:

A student learns a list of vocabulary words with an initial performance score of $P_0 = 100$. If the forgetting constant for this type of task is $c = 0.2$, what is the predicted performance score P after 3 months?

Solution:

Given $P_0 = 100$, $c = 0.2$ and $t = 3$ using $\log P = \log P_0 - c\log(t+1)$ we have:

$\log P = \log 100 - 0.2\log(3+1)$

$\log P = 2 - 0.2\log(4)$

$\log P = 2 - 0.2(0.60206)$

$\log P = 1.8796$

$P = 10^{1.8796} \approx 75.86(\text{out of } 100)$

Wealth Distribution

Vilfredo Pareto (1848–1923) observed that most of the wealth of a country is owned by a few members of the population. Pareto's Principle is $\log P = \log c - k\log W$ (or the 80-20 rule) where:

- W is the wealth level (how much money a person has);
- P is the number of people in the population having that much money;
- c is a scaling factor related to the total population;
- k(which is positive) is the Pareto index and determines how unequal the distribution is. A higher k means the wealth is concentrated among a smaller elite.

Example 11:

In a certain country, the wealth distribution follows the formula $\log P = 10 - 1.5\log W$.

How many people in this country have at least \$10,000? Give your answer as a whole number.

Solution:

$\log P = 10 - 1.5\log(10000)$

$\log P = 10 - 1.5(4)$

$\log P = 10 - 6 = 4 \rightarrow P = 10^4 = 10000\,\text{people}$

Interests: Simple, Compounded, Continuously

Simple:
Interest only on the original money. (Linear growth)
$I = \mathrm{P}\,\mathrm{r}\,t$ where P is the principal (amount of money, I interest amount, r rate and t is time

Compounded:
Interest on the original money **plus** past interest.
$P = P_0\left(1+\frac{r}{n}\right)^{nt}$ where P_0 initial amount invested, r is the rate, n is the number of times the interest is compounded in one year, and t is the time.

Continuously:
Interest added every split-second to maximize the "snowball" effect. (The fastest exponential growth).
$P = P_0 e^{rt}$

Example 12:

You invest $5,000 in an account that pays 6% annual interest, compounded continuously. How much will you have after 10 years?

Solution:

Using $P(t) = P_0 e^{rt}$:

$P(10) = 5000e^{6\%(10)} = 5000e^{0.6} = 5000(1.82) \approx \$9110.59$

Example 13:

A country's GDP grows at a constant rate of 3.5% per year. How many years will it take for the GDP to double?

Solution:

Using $2P_0 = P_0 e^{rt} \rightarrow 2 = e^{rt}$

$e^{0.35t} = 2$

$0.35t = \ln 2$

$t = \frac{\ln 2}{0.035} = \frac{0.6931}{0.035} \approx 19.8 \text{ years}$

Example 14:

Solve: $\log_2 x + \log_4 x + \log_8 x = \frac{11}{6}$

Solution:

$\log_2 x + \log_4 x + \log_8 x = \frac{11}{6} \rightarrow \log_2 x + \frac{\log_2 x}{\log_2 4} + \frac{\log_2 x}{\log_2 8} = \frac{11}{6} \rightarrow \log_2 x + \frac{\log_2 x}{2} + \frac{\log_2 x}{3} = \frac{11}{6}$

$\frac{6\log_2 x}{6} + \frac{3\log_2 x}{6} + \frac{2\log_2 x}{6} = \frac{11}{6}$

$\log_2 x = 1$

$x = 2$

Example 15:

Solve: $(\log_3 x)^2 - 3\log_3 x^2 + 8 = 0$

Solution:

$(\log_3 x)^2 - 3\log_3 x^2 + 8 = 0$

$(\log_3 x)^2 - 6\log_3 x + 8 = 0$

Let $u = \log_3 x$ then

$u^2 - 6u + 8 = 0$

$(u-4)(u-2) = 0 \rightarrow u = 2,\ u = 4$

Since $u = \log_3 x$ we have

$\log_3 x = 2,\ \ \log_3 x = 4 \rightarrow 3^2 = 9,\ \ 3^4 = 81$

Example 16:

Solve the system for x and y, where $x, y > 0$: $\begin{cases} \log_2 x + \log_4 y = 3 \\ \log_4 x + \log_2 y = \dfrac{9}{2} \end{cases}$

Solution:

$$\begin{cases} \log_2 x + \log_4 y = 3 \\ \log_4 x + \log_2 y = \dfrac{9}{2} \end{cases} \rightarrow \begin{cases} \log_2 x + \dfrac{\log_2 y}{\log_2 4} = 3 \\ \dfrac{\log_2 x}{\log_2 4} + \log_2 y = \dfrac{9}{2} \end{cases} \rightarrow \begin{cases} 2\left(\log_2 x + \dfrac{1}{2}\log_2 y\right) = (3)2 \\ \dfrac{1}{2}\log_2 x + \log_2 y = \dfrac{9}{2} \end{cases} \rightarrow$$

$$\rightarrow \begin{cases} 2\log_2 x + \log_2 y = 6 \\ -\left(\dfrac{1}{2}\log_2 x + \log_2 y = \dfrac{9}{2}\right) \end{cases} \rightarrow \frac{3}{2}\log_2 x = \frac{3}{2} \rightarrow \log_2 x = 1 \rightarrow x = 2$$

Solving for y: $\log_2 2 + \log_4 y = 3 \rightarrow \log_4 y = 2 \rightarrow y = 4^2 = 16$

Example 17:

A piano piece is learned with $P_0 = 90$. The constant for motor skills is $c = 0.15$. After how many months will the performance drop to $P = 60$? (round to the nearest integer)

Solution:

Using $\log P = \log P_0 - c\log(t+1)$ we have:

$\log 60 = \log 90 - 0.15\log(t+1)$

$$\frac{15}{100}\log(t+1) = \log 90 - \log 60 \rightarrow \frac{3}{20}\log(t+1) = \log\frac{90}{60} \rightarrow \frac{3}{20}\log(t+1) = \log\frac{3}{2} \rightarrow$$

$$\rightarrow \log(t+1) = \frac{20}{3}\left(\log\frac{3}{2}\right) \rightarrow \log(t+1) = \frac{20}{3}(0.1761) \rightarrow \log(t+1) = 1.174 \rightarrow$$

$t = 10^{1.174} - 1 = 13.93 \approx 14$ months

Example 18:

A researcher knows that after 6 months, performance on a typing test is $P = 45$ $P = 45$, and after 12 months, it is $P = 30$. Find the values of P_0 and c for this task. Then predict the performance after 24 months.

Solution:

Using $\log P = \log P_0 - c\log(t+1)$ we have:

$\log 45 = \log P_0 - c\log(6+1)$ and $\log 30 = \log P_0 - c\log(12+1)$.

Rewrite:

$\log 45 = \log P_0 - c\log(7)$ $eq.1$

$\log 30 = \log P_0 - c\log(13)$ $eq.2$

Subtract eq. 2 from eq. 1:

$\log 45 - \log 30 = -c\log(7) + c\log(13)$

$$\log\frac{45}{30} = c\left(\log\frac{7}{13}\right) \rightarrow c = \frac{\log\frac{45}{30}}{\left(\log\frac{13}{7}\right)} = \frac{0.1761}{0.2688} = 0.655$$

Solving for P_0:

$\log 45 = \log P_0 - 0.655(0.8451) \rightarrow \log P_0 = 1.6532 + 0.554 = 2.2072 \rightarrow P_0 = 10^{2.207} \approx 161$

After 24 months:

$\log P = \log 161 - 0.655\log(24+1)$

$\log P = 2.207 - 0.655(1.398) = 1.29131$

$P = 10^{1.29131} = 19.55$

Example 19:

You have \$8,000 now and want to grow it to \$12,000. Your account earns 7% annual interest, compounded quarterly. How many years will it take?

Solution:

Using $P = P_0\left(1+\frac{r}{n}\right)^{nt}$:

$$12000 = 8000\left(1+\frac{0.07}{4}\right)^{4t} \rightarrow \frac{12000}{8000} = (1.0175)^{4t} \rightarrow \frac{3}{2} = (1.0175)^{4t}$$

Taking ln on both sides:

$$\ln\left(\frac{3}{2}\right) = \ln(1.0175)^{4t}$$

$$0.405 = 4t\ln(1.0175)$$

$$t = \frac{0.405}{4\ln(1.0175)} = 5.84 \text{ years}$$

Example 20:

A small nation's wealth follows the rule $\log P = 8 - 2\log W$. If only 1 person in this nation has at least W dollars (i.e., $P = 1$), what is the minimum wealth W of that richest person? Solve for W in terms of log and then simplify to a whole number.

Solution:

$\log 1 = 8 - 2\log W$

$0 = 8 - 2\log W$

$\log W = 4 \rightarrow W = 10^4 = \$10000$

Example 21:

The country of Aurelia has a wealth distribution given by $\log P = \log c - k \log W$. If there are 100 people who have at least \$1,000 and only 1 person who has at least \$10,000,000, determine the value of the Pareto index k.

Solution:

$\log P = \log c - k \log W$

For P = *100* and W = *1000* we have: $\log 100 = \log c - k \log 1000 \rightarrow 2 = \log c - 3k$

For P = *1* and W = *10,000,000* we have: $\log 1 = \log c - k \log 10000000 \rightarrow 0 = \log c - 7k \rightarrow 7k = \log c$

Substitute $7k = \log c$ into $2 = \log c - 3k$ to have:

$2 = 7k - 3k \rightarrow k = \frac{1}{2}$

Practice Problems

Solve each equation.

1) $20^{8x} - 4 = 25$ $\frac{\log_{20} 29}{8}$

2) $-4 \cdot 12^{-10v} = -91$ $-\frac{\log_{12} \frac{91}{4}}{10}$

3) $17^{b+1} - 6 = 44$
$\log_{17} 50 - 1$

4) $9^{n-10} - 2 = 92$
$\log_{9} 94 + 10$

5) $-9e^{-7n-3} - 7 = -47$ $\frac{-\ln \frac{40}{9} - 3}{7}$

6) $-7e^{8-5m} - 3 = -76$ $\frac{-\ln \frac{73}{7} + 8}{5}$

7) $-10e^{-9x-10} + 3 = -60$ $\frac{-\ln \frac{63}{10} - 10}{9}$

8) $-9e^{-10x-5} + 6 = -73$ $\frac{-\ln \frac{79}{9} - 5}{10}$

9) $\log_{19}(-4b) = \log_{19}(8 - 2b)$
$\{-4\}$

10) $\log_{7}(-3r - 10) = \log_{7}(-2r - 5)$
$\{-5\}$

11) $\log_{2}(-4x + 5) = \log_{2}(5x - 4)$
$\{1\}$

12) Solve: $\log_{n}(2n - 3) = 1$
3

13) $\log_{n+1} 2 = -1$ $-\frac{1}{2}$

14) Solve :$\log_{n^2-n}(-2n) = 1$
-1

-

Solve each equation.

15) $\log_{2} p^2 = \log_{2}(40 - 3p)$
$\{-8, 5\}$

16) $\log_{6}(24 + 2r) = \log_{6}(r^2 - 3r)$
$\{-3, 8\}$

17) $\log_{2}(6v - 2) = \log_{2}(v^2 - 42)$
$\{10\}$

18) $\log_{5}(p^2 + 5p) = \log_{5}(-6 - 2p)$
$\{-6\}$

19) $\log_{11}(16v - 3) = \log_{11}(v^2 + 57)$
$\{10, 6\}$

20) $6\log_{5}(m + 7) = 12$
$\{18\}$

21) $9 + \log_{2} 8x = 10$ $\left\{\frac{1}{4}\right\}$

22) $-8\log_{2} 9x = -16$ $\left\{\frac{4}{9}\right\}$

23) $\log_{5}(-7n + 3) = 2$ $\left\{-\frac{1}{35}\right\}$

24) $-6\log_{6}(-4v - 4) + 4 = -2$ $\left\{-\frac{5}{2}\right\}$

25) $-2\log_{4}(1 - 9x) + 8 = 10$ $\left\{\frac{1}{12}\right\}$

26) $-2\log_{2}(2x - 1) + 5 = 7$ $\left\{\frac{3}{4}\right\}$

27) $-4\log_{3}(8n + 7) + 8 = 8$ $\left\{-\frac{3}{4}\right\}$

28) $\log_{5}(x + 8) - 4 = -3$
$\{-3\}$

29) $-10 + \log 6p = -9$ $\left\{\frac{5}{3}\right\}$

30) $3 + \log_{11}(n - 10) = 2$ $\left\{\frac{111}{11}\right\}$

31) $5\log_{12} -3n = 15$

$\{-576\}$

32) $\ln 10 - \ln(2-4x) = 5$ $\left\{\frac{-5+e^5}{2e^5}\right\}$

33) $\ln 10 + \ln(7-3x^2) = \ln 40$

$\{1,-1\}$

34) $\ln 10 - \ln(2-3x) = \ln 9$ $\left\{\frac{8}{27}\right\}$

35) $\ln(9-4x^2) + \ln 8 = \ln 54$ $\left\{\frac{3}{4}, -\frac{3}{4}\right\}$

36) $\ln 3 - \ln(6-2x) = \ln 13$ $\left\{\frac{75}{26}\right\}$

37) Solve: $\log_{\sqrt{2}}\left(\log_2\left(\log_4 x\right)\right) = 2$ [256]

38) Solve: $\log_2\left(\sqrt{x\sqrt{x\sqrt{x\sqrt{x.....}}}}\right) = 5$ [32]

39) $\begin{cases}\log_2 x - \log_2 y = 2\\ x - 2y = 1\end{cases}$ $\left[\left(2, \frac{1}{2}\right)\right]$

40) $\begin{cases}\log_3(x-y) = 1\\ \log_3 x + \log_3 y = 2\log_3 2\end{cases}$ $[(4,1)]$

41) $\begin{cases}\log_3(2x-y) = 0\\ 3^y + 3^x - \frac{4}{3} = 0\end{cases}$ $[(0,-1)]$

42) $\begin{cases}4\log_2 x - \log_2 y^2 = 4\\ \log_2 x + \log_2 y = 4\end{cases}$ $[(4,4)]$

43) Task A has $c_A = 0.25$ and Task B has $c_B = 0.10$ .If both start at $P_0 = 100$, find the time t at which Task A's performance is exactly half of Task B's performance. Solve for t exactly in terms of logarithms, then give a numerical approximation. [100.6 months]

44) An economist states that the wealth distribution in a city fits the model $\log P = 12 - 2.3\log W$. Show algebraically that the number of people who have at least \$1,000,000 is exactly $10^{2.3}$ times smaller than the number of people who have at least \$100,000.
(Hint: Set up the expression for $P(1{,}000{,}000)$ and $P(100{,}000)$, then divide them. Use the properties of logarithms to simplify the exponent.)

45) Emma invests \$5,000 in a savings account that earns simple interest at an annual rate of 4%.
(a) How much interest will she earn after 3 years? [600]
(b) What will be the total amount in her account at the end of this period? [5600]

46) A company invests \$10,000 in a fund that offers a 4% annual interest rate, compounded quarterly. How long will it take for the investment to grow to at least \$15,000? [11 years]

CHAPTER 2-Review

1. The function $f(x) = k(b^x) + c$ passes through points $(0,-5)$ and $(1,-11)$, and as $x \to \infty$ $y \to -3$. Write the equation of $f(x)$. $\left[y = -2\left(4^x\right) - 3\right]$
2. The function $f(x) = k(b^x) + c$ has a horizontal asymptote at $y = 1$. It passes through the y-intercept $\left(0,\frac{7}{6}\right)$ and the point $\left(1,\frac{3}{2}\right)$. If $b = 3$, find k and c. $\left[k = \frac{1}{6}; c = 1\right]$
3. The function $f(x) = k\left(b^{ax}\right) + c$ has a horizontal asymptote at $y = 20$. The graph passes through the y-intercept at (0, 12), and two other points: (- 2, -108) and (- 1, - 12). If the base is known to be $b = \frac{1}{2}$, find the exact values of k, a and c to write the equation of $f(x)$. $\left[y = -8\left(\frac{1}{2}\right)^{2x} + 20\right]$
4. The function $f(x) = k(b^x) + c$ passes through points $(0, 14)$ and $(-1,6)$, and as $x \to \infty$, $y \to 4$. Write the equation of $f(x)$. $\left[y = 10(5)^x + 4\right]$
5. Find the logarithmic equation $y = a\log_3(x - h)$ whose vertical asymptote is $x = -2$ and which passes through the point $(7, 6)$. $\left[y = 3\log_3(x + 2)\right]$
6. Given $y = a\log_b(x)$ which passes through the points $(b, 7)$, (b^2, d), and $(64, 21)$, find the values of a, b, and d. $[a = 7, b = 4, d = 14]$
7. A logarithmic function is defined by the equation $\log_5(x - a) + b$. The graph has a vertical asymptote at $x = -4$ and passes through the point $(6, 2)$. Find the exact values of a and b. $[a = -4, b = 2 - \log_3 10]$
8. Given $y = a\log_b(x)$ which passes through the points $(b, 4)$, (b^2, e), and $(27, 12)$, find the values of a, b, and e. $[a = -4, b = 3, e = 8]$
9. Solve $32\left(8^{3x-3}\right) = 32$ $[x = 1]$
10. Solve $2^x = 16\sqrt{2}$ $\left[x = \frac{9}{2}\right]$
11. Solve $3^{\sqrt{x+2}} = 9^{\sqrt{x}}$ $\left[x = \frac{2}{3}\right]$
12. Solve $3^{x+\frac{1}{2}} - 3^x = 9\left(\sqrt{3} - 1\right)$ $[x = 2]$
13. Solve $\sqrt{4^x} = \frac{1}{\sqrt{8}}$ $\left[x = -\frac{9}{4}\right]$
14. Solve $a^x\left(a^{2x+1}\right) = \frac{a^2}{\sqrt{a}}; a > 0$ $\left[a = \frac{9}{2}\right]$
15. Solve $4^x + 4^{(x-1)} + 4^{(x-2)} = 84$ $[x = 3]$
16. Solve $5^x + 5^{x+1} = 3^x + 3^{x+2}$ $[x = 1]$
17. Solve: $2^{2x} - 5\left(2^{x+1}\right) + 24 = 0$ $[x = 2, x = \log_2 6]$
18. Solve: $3^{2x+1} + 9 = 28\left(3^x\right)$ $[x = -1, x = 2]$
19. Solve: $2^{2x+2} - 2^{x+1} + 2^{x-1} = 20$ $[x = 3]$

20. Find the domain of $f(x)=\log_2\left(\frac{\sqrt{x+2}}{\sqrt{x+5}}\right)$ $[(5,\infty)]$

21. Find the domain of $f(x)=\ln\left(\frac{\sqrt{x+1}}{\sqrt{x^2-4}}\right)$ $[(2,\infty)]$

22. Find the domain of $f(x)=\log_3\left(\frac{x^2-5x+6}{x^2-4}\right)$ $[(-\infty,-2)\cup(2,3)\cup(3,\infty)]$

23. Find the domain of $f(x)=\ln\left(\ln\left(x^2-6x+10\right)\right)$ $[x\neq 3]$

24. Given $\log_9 11=1.1;\ \log_9 8=0.9;\ \log_9 6=0.8$; find $\log_9\frac{32}{27}$ $[0]$

25. Given $\log_5 7=1.2;\ \log_5 12=1.5;\ \log_5 8=1.3$; find $\log_5\frac{50}{3}$ $[1.8]$

26. Given $\log_4 6=1.3;\ \log_4 7=1.4;\ \log_4 9=1.6$; find $\log_4\frac{49}{16}$ $[0.8]$

27. Given $\log_7 10=1.2;\ \log_7 12=1.3;\ \log_7 9=1.1$; find $\log_7\frac{1}{441}$ $[-3.1]$

28. Given $\log_7 4=R;\ \log_7 10=S;\ \log_7 6=T$; find $\log_7\frac{1}{280}$ $[-1-S-R]$

29. Given $\log_5 11=R;\ \log_5 4=S;\ \log_5 6=T$; find $\log_5\frac{216}{5}$ $[3T-1]$

30. Given $\log_8 12=X;\ \log_8 9=Y;\log_8 5=Z$; find $\log_8\frac{1}{360}$ $[-1-Y-Z]$

31) Let $f(x)=3\ln x$ and $g(x)=\ln(5x^3)$
Express $g(x)$ in the form $f(x)+\ln a$, where a is a positive integer.

$a=5$

32) Let $f(x)=\log_3\frac{x}{2}+\log_3 16-\log_3 4$ for $x>0$. Show that the $f(x)$ can be written in the form $f(x)=\frac{\ln ax}{\ln b}$. Find the value of a and b

$a=2,\ b=3$

33) If $6^a=2,\ 6^b=5$ and $6^c=15$ then express c in terms of a and/or b

$b-a+1$

34) Solve for x:
$\log_{27}(\log_2(x+1))=\frac{1}{3}$

9

35) Solve:
$1+2\log_4(x+1)=2\log_4(x+2)$

0

36) Solve:
$\log_4 2^x=5x+3$

$-\frac{2}{3}$

37) If $\log 8+\log(4x-1)=\log(2x+4)+1$, then the value of $x+7$ is

11

38) Given that $\log_2\frac{32^x}{8^y}$ can be written as $px+qy$, find the value of p and q

$p=5,\ q=3$

39) Given that $p=\log_a 5$ and $q=\log_a 2$, express $\log_a 2.5$ in terms of p and/or q.

$p-q$

40) Solve:
$\log_3(3^{2x}-3^x+1)=x$

0

41) If $\log_4 2^x = 5x + 3$ has one solution $x = a$, find $a + 3$ $\frac{7}{3}$

42) Solve: $\log_2 (2x + 6) - \log_4 (x - 1) = 3$
5

43) Solve: $2 \cdot (\log_2 x)^2 + \log_2 x^5 - 3 = 0$ $\frac{1}{8}; \sqrt{2}$

44) Solve: $\frac{3}{\ln x} + \frac{\ln x}{\ln x + 1} = 2 + \frac{1}{\ln x}$
$e^{\sqrt{2}}; e^{-\sqrt{2}}$

45) Show that $\log_{a^2} (a\sqrt{a}) = \log_a \sqrt[4]{a^3}$

46) Solve: $\left(\frac{2}{5}\right)^{x+3} = \left(\frac{5}{2}\right)^{x-2}$ $x = -\frac{1}{2}$

47) Solve: $17\sqrt{2^{x+1}} = 34\sqrt[3]{4^{x-3}}$
$x = 9$

48) Solve: $-4^x - 3 \cdot 2^x = 2^{2x} - 2^x$
no solution

49) Solve: $\log_2 (x^2 - 1) = 3$
$x = -3$ or $x = 3$

50) Solve: $\log_{\frac{1}{\sqrt{7}}} (16 - 9x^2) = -2$
$x = -1, \ x = 1$

51) Solve: $\frac{1}{2} \cdot \log (x - 2x^2) = \log x$ $x = \frac{1}{3}$

52) Solve: $x^{\log_2 x} = 16x^3$ $x = 16, x = \frac{1}{2}$

53) $\log_x 4 + \log_x 8 = 5$
$x = 2$

54) Solve:
$\log_x (2x^2 - 3x + 2) = 2 + \log_x (x - 1)$
$x = 2$

Find the inverse of each function.

55) $y = \left(\frac{e^x - 3}{3}\right)^{\frac{1}{2}}$
$y = \ln (3x^2 + 3)$

56) $y = \left(\frac{e^x + 7}{-3}\right)^{\frac{1}{5}}$
$y = \ln (-3x^5 - 7)$

57) $y = \left(\frac{e^x - 1}{-4}\right)^{\frac{1}{3}}$
$y = \ln (-4x^3 + 1)$

58) $y = \left(\frac{e^x + 2}{3}\right)^{\frac{1}{4}}$
$y = \ln (3x^4 - 2)$

Solve each continuous exponential growth/decay problem.

59) For a period of time, an island's population grows exponentially. If the continuous growth rate is 3% per year and the current population is 1,551, what will the population be 5 years from now?
1,802

60) Atmospheric pressure decreases approximately exponentially as elevation increases. If the continuous rate of decrease is determined to be 12% per kilometer and the pressure at sea level is 760.0 mmHg, what is the pressure at 5 kilometers above sea level?

61) For a period of time, an island's population grows exponentially. If the population was 990 8 years ago and the current population is 1,363, what will the population be 5 years from now?

$$1363 \cdot \left(\frac{1363}{990}\right)^{\frac{5}{8}} \approx 1{,}664$$

62) For a period of time, E. coli bacteria in a culture grows exponentially. If the population was 78.4 million 8 minutes ago and the current population is 108.0 million, what will the population be 7 minutes from now?

$$108 \cdot \left(\frac{108}{78.4}\right)^{\frac{7}{8}} \approx 142.9 \text{ million}$$

63) For a period of time, E. coli bacteria in a culture grows exponentially. If the population was 86.9 million 7 minutes ago and the current population is 100.0 million, what will the population be 5 minutes from now?

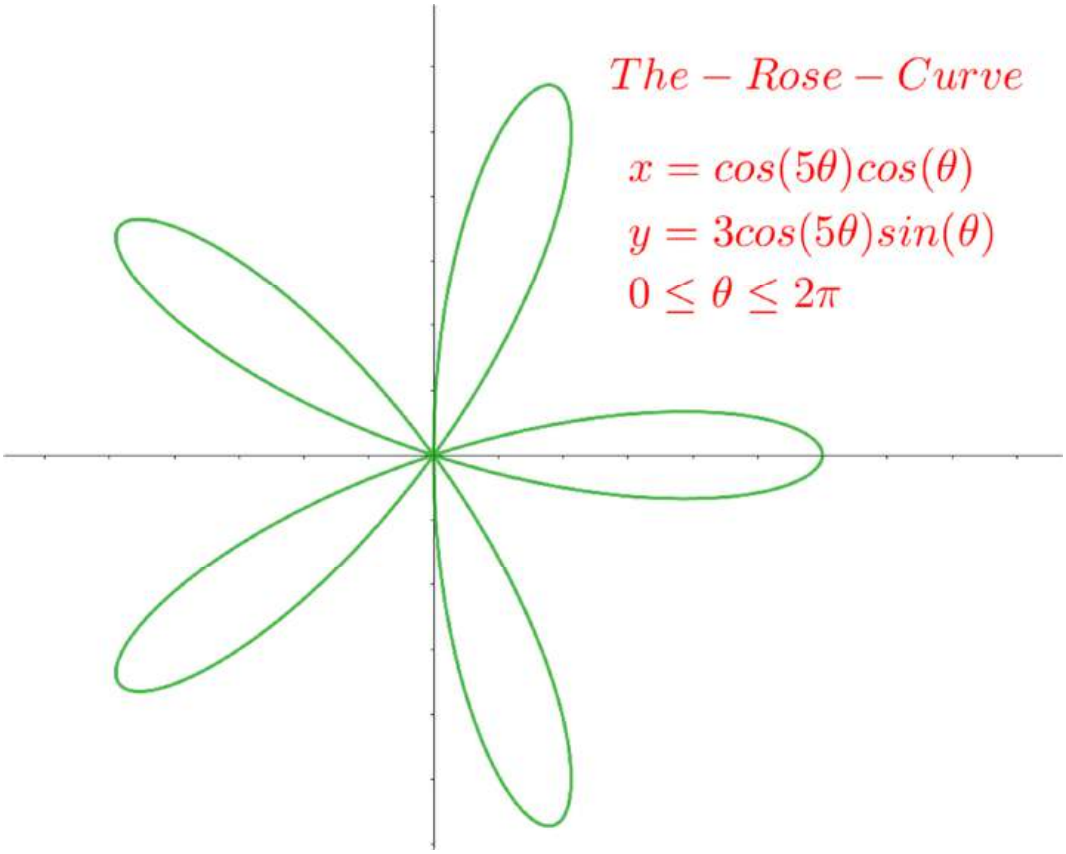
$$100 \cdot \left(\frac{100}{86.9}\right)^{\frac{5}{7}} \approx 110.5 \text{ million}$$

64) For a period of time, an island's population grows exponentially. If the population was 1,631 5 years ago and the current population is 1,895, what will the population be 10 years from now?

$$1895 \cdot \left(\frac{1895}{1631}\right)^{\frac{10}{5}} \approx 2{,}558$$

Chapter 3 – Trigonometric Functions

$e^{i\theta} = \cos\theta + i\sin\theta$ *"This is our jewel... the most remarkable formula in mathematics." — Richard Feynman*



Review

1. Given $f(x) = -x^2 + 3x - (c+1)$ find the value(s) of c so $f(x)$ has one real root. $\left[c = \frac{5}{4}\right]$

2. Given $f(x) = 2x^2 - 4x + m$ find the value(s) of m so $f(x)$ has no real root. $[m > 2]$

3. Write the vertex form of the parabola $y = -3(x-10)(x+3)$ $\left[y = -3\left(x - \frac{7}{2}\right)^2 + \frac{507}{4}\right]$

4. A rectangular field with area of 247 m^2 is enclosed by 64m of fencing. Find the dimension of the field. $[13 \times 19]$

3.1 Angles and their Measurement

Topic: Angle Measurement, Coterminal and Reference Angles, Right Triangle Definitions, Trigonometric Functions and their Inverses.

Definition of an angle

An angle is a part of a plane formed by two rays that share a common endpoint called the **vertex**. The two rays are referred to as the **initial side** and the **terminal side**.

An angle is said to be in **standard position** when:

1. Its vertex is located at the origin of a coordinate system

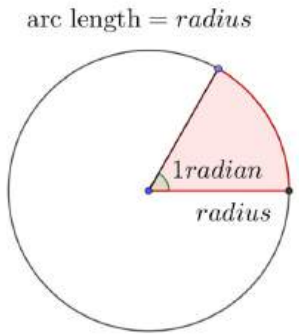
2. Its initial side lies along the positive x-axis

In standard position, the ray on the x-axis is called the initial side, and the other ray is called the terminal side.

Angle Measurement

Angles can be measured in two primary units: *degrees and radians*.
Degrees: A degree is a unit for measuring angles where one full rotation (complete circle) is divided into 360 equal parts. Each degree represents $\frac{1}{360}$ of a full rotation.
Radians: A radian is a unit based on the radius of a circle. One radian is the angle created when the arc length along the circle is equal to the radius. In radian measure, a full circle equals 2π radians.

Converting between Degrees and Radians



To convert between degrees and radians, use the following relationships:

From degrees to radians: multiply the number of degrees by $\frac{\pi}{180^\circ}$

From radians to degrees: multiply the number of radians by $\frac{180^\circ}{\pi}$

Example 1:

Convert 45° to radians
Solution:

$$45^\circ \frac{\pi}{180^\circ} = \frac{\pi}{4} \text{ radians}$$

Example 2:

Convert 50° to radians
Solution:

$$50^\circ \frac{\pi}{180^\circ} = \frac{5}{18}\pi \text{ radians}$$

Example 3:

Convert $\frac{5}{4}\pi$ radians to degrees

Solution:

$$\frac{5}{4}\pi\frac{180^\circ}{\pi}=225^\circ$$

Example 4:

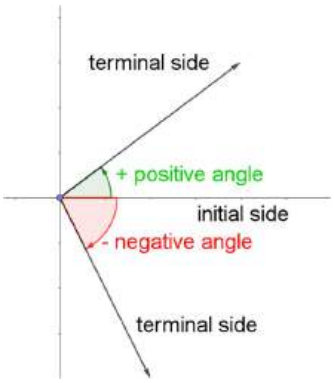
Convert $\frac{5}{6}\pi$ radians to degrees

Solution:

$$\frac{5}{6}\pi\frac{180^\circ}{\pi}=150^\circ$$

Positive -Negative Angles

Angles can be positive or negative. We consider a *counterclock* rotation to be *positive* and a *clockwise* rotation to be *negative*.



Quadrants

The coordinate plane is divided into four quadrants:

Quadrant 1: from 0° to 90° or from 0 to $\frac{\pi}{2}$

Quadrant 2: from 90° to 180° or from $\frac{\pi}{2}$ to π

Quadrant 3: from 180° to 270° or from π to $\frac{3}{2}\pi$

Quadrant 4: from 270° to 360° or from $\frac{3}{2}\pi$ to 2π

Example 5:

State the quadrant of 96°

Solution:

$90^\circ<96^\circ<180^\circ\rightarrow$ Quadrant 2 .

Example 6:

State the quadrant of 238°

Solution:

$90^\circ<^\circ<180^\circ\rightarrow$ Quadrant 2 .

Example 7:

State the quadrant of $\frac{5}{3}\pi$

Solution:

$\frac{2}{3}\pi<\frac{5}{3}\pi<2\pi\rightarrow$ Quadrant 4 .

Example 8:

State the quadrant of $\frac{5}{6}\pi$

Solution:

$\frac{\pi}{2}<\frac{5}{6}\pi<\pi\rightarrow$ Quadrant 2 .

Coterminal Angles

The coterminal angles are the angles that have the same initial side and the same terminal sides. We determine the coterminal angle of a given angle by adding or subtracting 360° or 2π to it. For any given angles there are infinitely many coterminal angles. The measures of all angles coterminal with a k° angle are given by the expression $k^\circ + n360^\circ$, where n is an integer.

Example 9:

Find a positive and negative coterminal angles of 250°

Solution:

Positive: $250^\circ + 360^\circ = 610^\circ$

Negative: $250^\circ - 360^\circ = -110^\circ$

Example 10:

Find a positive and negative coterminal angles of $\frac{3}{4}\pi$

Solution:

Positive: $\frac{3}{4}\pi + 2\pi = \frac{11}{4}\pi$

Negative: $\frac{3}{4}\pi - 2\pi = -\frac{5}{4}\pi$

Reference Angles

The *reference angle* is the smallest positive angle formed by the terminal side of the given angle and the *x*-axis. It is always an acute angle (between 0° and 90°) except when the angle is exactly 90°.

Rules For Finding Reference Angles

Quadrant	Angle Range	Reference Angle
I	$0^\circ < \theta < 90^\circ$ or $0 < \theta < \frac{\pi}{2}$	θ
II	$90^\circ < \theta < 180^\circ$ or $\frac{\pi}{2} < \theta < \pi$	$180^\circ - \theta$ or $\pi - \theta$
III	$180^\circ < \theta < 270^\circ$ or $\pi < \theta < \frac{3}{2}\pi$	$\theta - 180^\circ$ or $\theta - \pi$
IV	$270^\circ < \theta < 360^\circ$ or $\frac{3}{2}\pi < \theta < 2\pi$	$360^\circ - \theta$ or $2\pi - \theta$

Example 11:

Find the reference angle of 260°

Solution:

260° is Quadrant 3 then

reference = $260^\circ - 180^\circ = 80^\circ$.

Example 12:

Find the reference angle of 140°

Solution:

140° is Quadrant 2 then

reference = $180^\circ - 140^\circ = 40^\circ$.

Example 13:

Find the reference angle of $\frac{11}{6}\pi$

Solution:

$\frac{11}{6}\pi$ is Quadrant 4 then

reference $= 2\pi - \frac{11}{6}\pi = \frac{1}{6}\pi$.

Example 14:

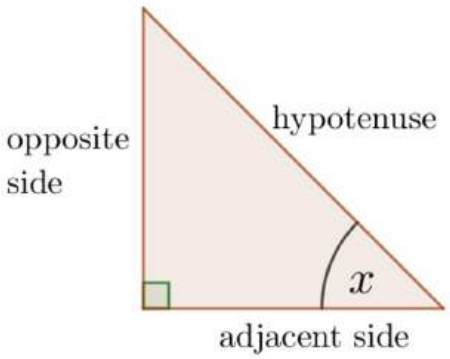
Find the reference angle of $\frac{7}{5}\pi$

Solution:

$\frac{7}{5}\pi$ is Quadrant 3 then

reference $= \frac{7}{5}\pi - \pi = \frac{2}{5}\pi$.

Right Triangle Definition



For a right triangle with angle x, the fundamental trigonometric functions are defined as:

$$\sin x = \frac{opposite}{hypotenuse} \quad \cos x = \frac{adjacent}{hypotenuse} \quad \tan x = \frac{opposite}{adjacent} = \frac{\sin x}{\cos x}$$

Where:

sine of $x = \sin x$; cosine of $x = \cos x$; tangent of $x = \tan x$

And the reciprocal functions are defined as:

$$\csc x = \frac{hypotenuse}{opposite} = \frac{1}{\sin x} \quad \sec x = \frac{hypotenuse}{adjacent} = \frac{1}{\cos x} \quad \cot x = \frac{adjacent}{opposite} = \frac{1}{\tan x}$$

where

cosecant of $x = \csc x$; secant of $x = \sec x$; cotangent of $x = \cot x$

TABLE OF VALUES OF FUNDAMENTAL FUNCTIONS AT FUNDAMENTAL ANGLES

x	0°	$30^\circ = \frac{\pi}{6}$	$45^\circ = \frac{\pi}{4}$	$60^\circ = \frac{\pi}{3}$	$90^\circ = \frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

NOTE:

When an angle is given, its trigonometric functions are uniquely determined. When, however, the value of one function of an angle is given, the angle is not uniquely determined. We can use a GDC by pressing 2nd and the SIN, COS or TAN. They are the inverse of the trigonometric function sine, cosine and tangent. The simbols for the inverses are $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ respectively for sine, cosine and tangent.

REMEMBER: $\sin^{-1} x$ is a number between -1 and 1 whose sine is x. The notation **is not** exponential notation. It does not mean $\frac{1}{\sin x}$.

FUNDAMENTAL IDENTITIES-PYTHAGOREAN IDENTITIES

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$

Example 15:

Find $\cot x$ if $\tan x = \frac{3\sqrt{19}}{95}; 0^\circ \le x \le 90^\circ$

Solution:

$\cot x = \frac{1}{\tan x}$ therefore $\frac{3\sqrt{19}}{95}$

Example 16:

Find $\csc x$ if $\sin x = \frac{\sqrt{5}}{5}; 0^\circ \le x \le 90^\circ$

Solution:

$\csc x = \frac{1}{\sin x}$ therefore $\frac{5}{\sqrt{5}}$

Example 17:

Find $\tan x$ if $\sec x = \frac{5}{4}$ $0^\circ \le x \le 90^\circ$

Solution:

$\sec x = \frac{1}{\cos x}$ therefore $\cos x = \frac{4}{5}$.

Using $\sin^2 x + \cos^2 x = 1$

We have $\sin x = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

Since $\tan x = \frac{\sin x}{\cos x} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

Example 18:

Find $\sec x$ if $\csc x = \frac{13}{12}$ $0^\circ \le x \le 90^\circ$

Solution:

$\sec x = \frac{1}{\cos x}$ therefore $\cos x = \frac{2}{13}$.

Using $\sin^2 x + \cos^2 x = 1$

We have $\sin x = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{\frac{25}{169}} = \frac{5}{13}$

Since $\csc x = \frac{1}{\sin x} = \frac{1}{\frac{5}{13}} = \frac{13}{5}$

Example 19:

Find $\sec x$ if $\cot x = \frac{2}{11}$ $0 \le x \le \frac{\pi}{2}$

Solution:

$\cot x = \frac{1}{\tan x}$ therefore $\tan x = \frac{11}{2}$.

Using $\tan^2 x + 1 = \sec^2 x$

We have $\sec x = \sqrt{1 + \left(\frac{11}{2}\right)^2} = \sqrt{\frac{125}{4}} = \frac{5\sqrt{5}}{2}$

Example 20:

Find $\csc x$ if $\cos x = \frac{24}{25}$ $0 \le x \le \frac{\pi}{2}$

Solution:

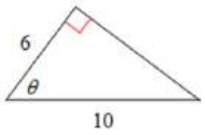
Using $\sin^2 x + \cos^2 x = 1$

We have $\sin x = \sqrt{1 - \left(\frac{24}{25}\right)^2} = \sqrt{\frac{49}{625}} = \frac{7}{25}$

$\csc x = \frac{1}{\sin x} = \frac{1}{\frac{7}{25}} = \frac{25}{7}$

Example 21:

Find the value of $\tan\theta$: $0 \le \theta \le \frac{\pi}{2}$



Solution:

Missing the opposite side (opp); by the Pythagorean theorem:

$\text{opp} = \sqrt{10^2 - 6^2} = 8$ therefore

$\tan x = \frac{opp}{adj} = \frac{8}{6} = \frac{4}{3}$.

Example 22:

Find the value of $\sec\theta$: $0 \le \theta \le \frac{\pi}{2}$

Solution:

Missing the hypotenuse(hypo); by the Pythagorean theorem:

$\text{hypo} = \sqrt{3^2 + 4^2} = 5$ therefore

$\sec x = \frac{hypotenuse}{adjacent} = \frac{5}{3}$.

Complementary Angles

Two angles are *complementary* if the sum of their measures is exactly 90 degrees. If one angle is 30°. its complement is 60° because $30+60=90$.

Supplementary Angles

Two angles are *supplementary* if the sum of their measures is exactly 180 degrees. If one angle is 110°, its supplement is 70° because $70+110=180$.

Example 23:

The measure of an angle is *3x + 10* degrees. Its complement is *2x - 5* degrees. Find the value of *x* and the measure of both angles.

Solution:

$(3x+10)+(2x-5)=90$

$5x+5=90$

$5x=85$

$x=17$

Then the two angle are 61° and 39°

Example 24:

The sum of the supplement of angle *A* and the complement of angle *A* is 190 degrees. Find the measure of angle *A*, complement and supplement of *A*.

Solution:

Let angle $A=x$

complement $A=90-x$

supplement $A=180-x$

Their sum is 190: $(180-x)+(90-x)=190$

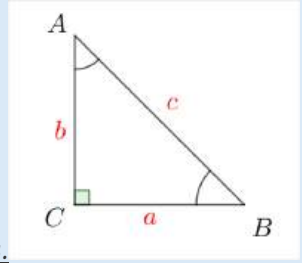
$270-2x=190$

$-2x=-80$

$x=40$

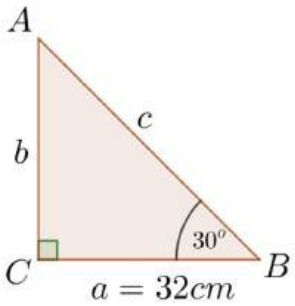
Therefore: angle $A=40^\circ$; complement $A=90-40=50^\circ$; supplement $A=180-40=140^\circ$

Any triangle ABC has six measurements associated with it: the lengths of the three sides, denoted by *a*, *b*, and *c* and the measures of the three angles, denoted by $\angle A$, $\angle B$, and $\angle C$. It is a good practice to draw the triangle. *The lower-case letters denote the lengths of sides opposite angles labeled by the corresponding capital letters.*



Example 25:

Find the measurements of the missing angles and sides :



Solution:

We use the definition of tangent of an angle

$tan\,\theta = \dfrac{opposite\ side\ to\ \theta}{adjacent\ side\ to\ \theta}$ then we have

$$\tan 30^{\circ} = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{b}{32}$$

$$b = 32 \times \tan 30^{0} = 32 \times \frac{1}{\sqrt{3}} = \frac{32}{\sqrt{3}}\,cm$$

Using Pythagorean Theorem to find c:

$$c = \sqrt{32^2 + \left(\frac{32}{\sqrt{3}}\right)^2} = \sqrt{1024 + \frac{1024}{3}} = \sqrt{\frac{4096}{3}} = \frac{64}{\sqrt{3}}\,cm$$

Since $\angle A$ and $\angle B$ are complementary:

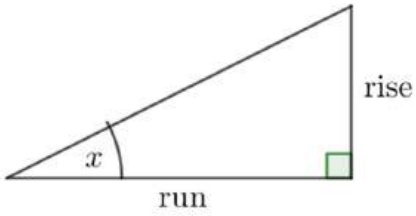
$$\angle A = 90 - 30 = 60^{0}$$

Example 26:

The grade of a road is the ratio of its rise to its run and is usually given as a decimal or percent. Find the angle (to the nearest degree) that the road makes with the horizontal if its grade is 57.73%.

Solution:

The problem can be visualized by the diagram below:

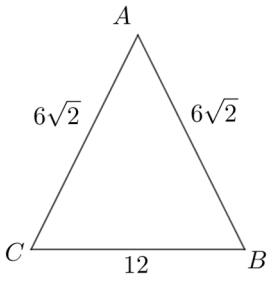


Since $57.73\% = 0.5773 = \dfrac{1}{\sqrt{3}}$ we have:

$$\tan x = \frac{1}{\sqrt{3}} \rightarrow x = 30^{\circ}$$

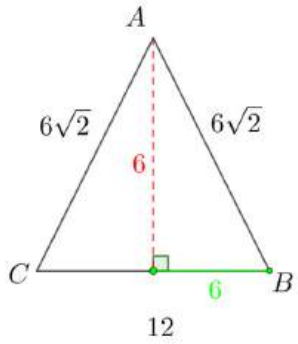
Example 27:

Find the measures (to the nearest degree)of the three angle of triangle ABC



Solution:

The triangle is an isosceles triangle so the altitude from angle A bisects BC and it is perpendicular to BC.



We can use the definition of cosine:

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{6}{6\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \text{ then}$$

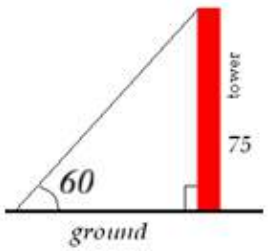
$$\cos B = \frac{\sqrt{2}}{2} \rightarrow B = 45^\circ \text{. Therefore } C = 45^\circ \rightarrow A = 90^\circ$$

Example 28:

A wire is attached to the top of a 75 meters tower and meets the ground at a 60º angle. How long (to the nearest meter) is the wire?

Solution:

The problem can be visualized by the diagram below:



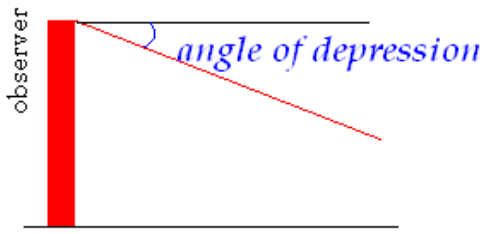
The wire is the hypotenuse of the right triangle. We can use the definition of the sine to have:

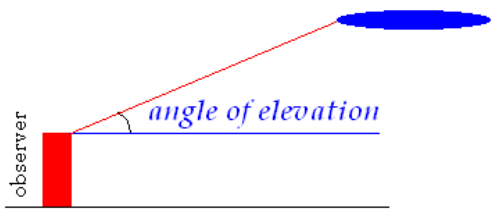
$$\sin 60^0 = \frac{\text{opposite}}{\text{wire}} \text{ then}$$

$$\text{wire} = \frac{\text{opposite}}{\sin 60^0} = \frac{75}{\frac{\sqrt{3}}{2}} = 50 \text{ meters}$$

The previous problem give us the opportunity to introduce two other definitions:

Angle of depression:	The angle by which an observer's line of sight must be lowered from the horizontal to see an object.
Angle of elevation:	The angle by which an observer's line of sight must be raised from the horizontal to see an object.



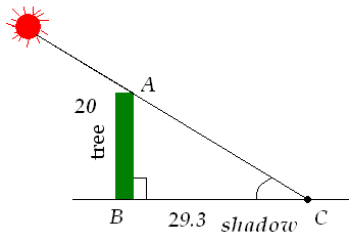


Example 29:

What is the angle of elevation of the sun (to the nearest degree)when a tree 20 meters tall casts a shadow 29.3 meters long?

Solution:

The problem can be visualized by the diagram below:

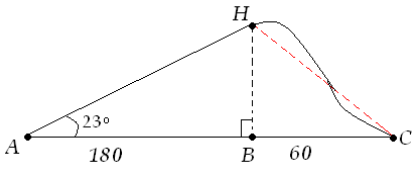


$\angle C$ is the angle of elevation. We can use the definition of tangent:

$$\tan C = \frac{\text{opposite}}{\text{adjacent}} = \frac{20}{29.3} = 0.6826 \text{ then}$$

Example 30:

A roller coaster takes 23.5 seconds to go up the 23° incline AH and only 2.8 seconds to go down the drop from H to C. The car covers horizontal distances of 180 meters on the incline and 60 meters on the drop.



(i) How high (to the nearest meter) is the roller coaster above point B?
(ii) Find the distance HC (to the nearest meter).
(iii) How fast in *m/s* does the car go up the incline?
(iv) What is the approximate average speed of the car as it goes down the drop? (assume the car travels along HC)

Solution:

(i) First we find AH by using cosine: $\cos 23^0 = \frac{180}{AH}$ then $AH = \frac{180}{\cos 23^0} = 195.545$. Now using Pythagorean Theorem: $HB = \sqrt{195.545^2 - 180^2} = \sqrt{5837.847} = 76.4 = 76 meters$

(ii) Again using Pythagorean Theorem: $HC = \sqrt{76^2 + 60^2} = \sqrt{9376} = 46.65 = 96.8 meters$

(iii) Since $\text{velocity} = \frac{\text{distance}}{\text{time}}$ we have $\text{velocity along } AH = \frac{76}{23.5} = 3.234 m/s$

(iv) In the same way $\text{velocity along } HC = \frac{96.8}{2.8} = 34.57 m/s$

Practice Problems

Convert each degree measure into radians and each radian measure into degrees.

1) 0 $0°$

2) -100° $-\frac{5\pi}{9}$

3) 570° $\frac{19\pi}{6}$

4) $\frac{17\pi}{6}$ $510°$

5) 405° $\frac{9\pi}{4}$

6) 225° $\frac{5\pi}{4}$

7) -60° $-\frac{\pi}{3}$

8) 135° $\frac{3\pi}{4}$

9) $\frac{7\pi}{6}$ $210°$

10) $-\frac{7\pi}{12}$ $-105°$

State the quadrant in which the terminal side of each angle lies.

11) 330° IV

12) −170° III

13) 140° II

14) 260° III

15) $\frac{11\pi}{9}$ III

16) $\frac{4\pi}{3}$ III

17) $\frac{5\pi}{18}$ I

18) $-\frac{11\pi}{12}$ III

Find a positive and a negative coterminal angle for each given angle.

19) −250° 110° and −610°

20) −195° 165° and −555°

21) 210°
570° and −150°

22) 690°
330° and −30°

23) 615°
255° and −105°

24) 400°
40° and −320°

25) 480°
120° and −240°

26) 60°
420° and −300°

Find the reference angle.

27) 255° 75°

28) 620° 80°

29) −510° 30°

30) −185° 5°

31) 490° 50°

32) $-\frac{11\pi}{4}$ $\frac{\pi}{4}$

33) $-\frac{11\pi}{6}$ $\frac{\pi}{6}$

34) $\frac{7\pi}{4}$ $\frac{\pi}{4}$

35) $-\frac{13\pi}{4}$ $\frac{\pi}{4}$

36) $-\frac{29\pi}{18}$ $\frac{7\pi}{18}$

37) $-520°$ $20°$

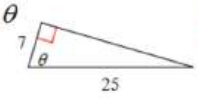
38) $300°$ $60°$

39) $-680°$ $40°$

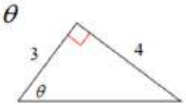
40) $\frac{11\pi}{12}$ $\frac{\pi}{12}$

Find the value of the trig function indicated.

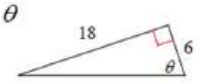
41) $\sin\theta$ $\frac{24}{25}$



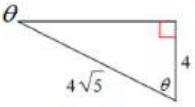
42) $\cos\theta$ $\frac{3}{5}$



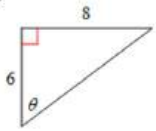
43) $\tan\theta$ 3



44) $\sin\theta$ $\frac{2\sqrt{5}}{5}$

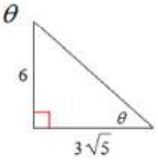


45) $\sin\theta$ $\frac{4}{5}$

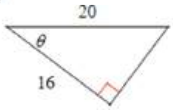


46) $\cot\theta$ $\frac{5}{12}$

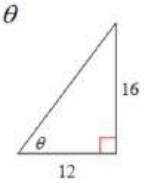
47) $\sec\theta$ $\frac{3\sqrt{5}}{5}$



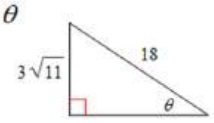
48) $\csc\theta$ $\frac{5}{3}$



49) $\cot\theta$ $\frac{3}{4}$



50) $\csc\theta$ $\frac{6\sqrt{11}}{11}$



51) Find $\cot\theta$ if $\sec\theta = \frac{17}{15}$ $\frac{15}{8}$

52) Find $\tan\theta$ if $\cot\theta = \frac{8}{15}$ $\frac{15}{8}$

53) Find $\tan\theta$ if $\cos\theta = \frac{4}{5}$ $\frac{3}{4}$

54) Find $\csc\theta$ if $\cot\theta = \frac{4}{3}$ $\frac{5}{3}$

55) Find $\tan\theta$ if $\sin\theta = \frac{3}{5}$ $\frac{3}{4}$

56) Find $\cot\theta$ if $\csc\theta = \frac{5}{3}$ $\frac{4}{3}$

57) Find $\tan\theta$ if $\cot\theta = \frac{3\sqrt{5}}{2}$ $\frac{2\sqrt{5}}{15}$

58) Find $\csc\theta$ if $\sec\theta = \frac{5}{4}$ $\frac{5}{3}$

59) Two angles are supplementary. One angle is 5 degrees less than four times the other. Find the measure of both angles.

$37°$ and $143°$

60) The difference between the measures of two complementary angles is 18 degrees. Find the measure of each angle.

$36°$ and $54°$

61) Angle A and angle B are supplementary. Angle A is $4x + 20$ degrees and angle B is $6x - 40°$. If angle C is complementary to angle A, find the measure of angle C.

$C=15°$

62) Angle P and angle Q are supplementary. Angle R and angle S are complementary. Given: $P = 5x - 20$, $Q = 3x + 40$, $R = 2y + 10$, $S = y - 10$. If $P = R$ find the values of x, y and all four angles

$x = 20$, $y = 20$, $P = R = 80$, $Q = S = 100$

3.2 Exploring Trigonometric Functions

Topics: Terminal side, Unit Circle, Domain, Range, Amplitude and Period

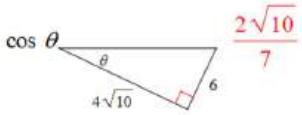
Review

1. Find a the reference angle:

   (a) $\frac{2}{3}\pi$ $\left[\frac{\pi}{3}\right]$

   (b) $\frac{13}{8}\pi$ $\left[\frac{3}{8}\pi\right]$

2. Find the value of the trig function indicated:



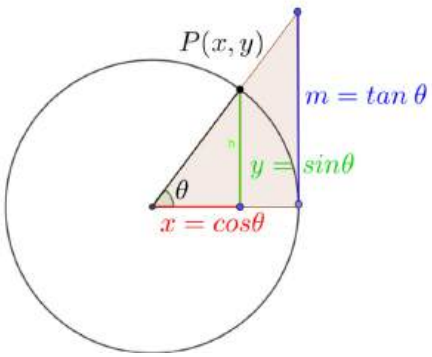
$\left[\frac{2\sqrt{10}}{7}\right]$

3. Find $\cot x$ if $\csc x = \frac{5}{4}$ $\left[\frac{3}{4}\right]$

4. Find $\sec x$ if $\csc x = \frac{25}{7}$ $\left[\frac{25}{24}\right]$

5. Angle A and angle B are supplementary. Angle U and angle V are complementary. Given $A = 5x+3, B = 9x+9, U = 3y, V = 5y+18$. If $A = V$ find the values of x, y and all four angles. $\left[x = 12, y = 9, A = 63^\circ, B = 117^\circ, U = 27^\circ, V = 63^\circ\right]$

Find Trigonometric Functions from Terminal Side



If the point in the angle's terminal side is $P = (x, y)$

then the trigonometric functions can be calculated as:

$$\sin\theta = \frac{y}{r},\quad \cos\theta = \frac{x}{r},\quad \tan\theta = \frac{y}{x}$$

where $r = \sqrt{x^2 + y^2}$

The Unit Circle (Special Case)

If the radius of the circle is 1 we have $x^2+y^2=1$. So the above formula simplify to:

$\sin\theta = y,\ \ \cos\theta = x,\ \ \tan\theta = \frac{y}{x}$.

This means:

The *x-coordinate* of a point on the unit circle is $\cos\theta$.

The *y-coordinate* is $\sin\theta$.

The tangent is the *slope* of the line from the origin to that point.

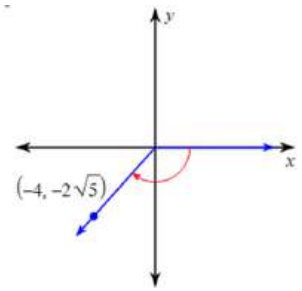
angle	0	$90^\circ = \frac{\pi}{2}$	$180^\circ = \pi$	$270^\circ = \frac{3}{2}\pi$
sin	0	1	0	-1
cos	1	0	-1	0
tan	0	undefined	0	undefined

Signs of Trig Functions (by quadrants)

Quadrant	sine	cosine	tangent
I	+	+	+
II	+	-	-
III	-	-	+
IV	-	+	-

Example 1:

Given point on the terminal side of angle *x*, find the value of sin *x*.



Solution:

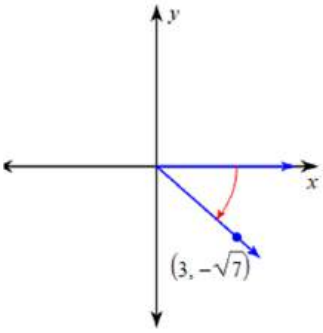
$$r=\sqrt{x^2+y^2}=\sqrt{(-4)^2+\left(-2\sqrt{5}\right)^2}=\sqrt{16+20}=6$$

The angle is on the third quadrant so the sine is negative.

Therefore: $\sin x=\frac{-2\sqrt{5}}{6}=-\frac{\sqrt{5}}{3}$

Example 2:

Given point on the terminal side of angle x, find the value of cot x.

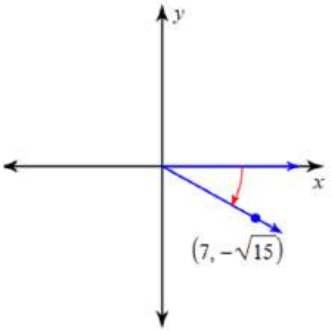


Solution:

The angle is on the fourth quadrant so the cotangent is negative. Therefore: $\cot x = \dfrac{3}{-\sqrt{7}} = -\dfrac{3\sqrt{7}}{7}$

Example 3:

Given point on the terminal side of angle x, find the value of csc x.



Solution:

$$r = \sqrt{x^2+y^2} = \sqrt{(7)^2+\left(-\sqrt{15}\right)^2} = \sqrt{49+15} = 8$$

The angle is on the fourth quadrant so the cosecant is negative. Therefore: $\csc x = \dfrac{8}{-\sqrt{15}} = -\dfrac{8\sqrt{15}}{15}$

Graphs and Domain and Range of Basic Trigonometric Functions

The domain of a function is the set of all possible input values (often x or θ), while the range is the set of all possible output values (often y). For trigonometric functions, these are determined by their definitions and the behavior of their graphs.

The table below summarizes the domain and range for the six basic trigonometric functions.

Functions	Domain	Range
sin x	$(-\infty,\infty)$	$[-1,1]$
cos x	$(-\infty,\infty)$	$[-1,1]$
tan x	$x \neq \dfrac{\pi}{2}+n\pi$	$(-\infty,\infty)$
csc x	$x \neq n\pi$	$(-\infty,-1]\cup[1,\infty)$
sec x	$x \neq \dfrac{\pi}{2}+n\pi$	$(-\infty,-1]\cup[1,\infty)$
cot x	$x \neq n\pi$	$(-\infty,\infty)$

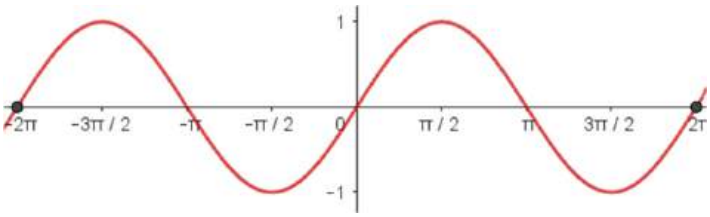
Notes on the Table:

n represents any integer (..., -2, -1, 0, 1, 2, ...). This accounts for the periodic nature of the functions.

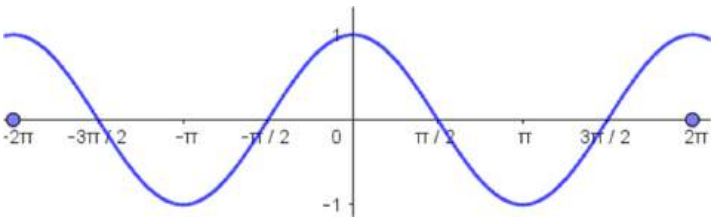
The domains for tangent and secant are the same, and the domains for cosecant and cotangent are the same.

The ranges for sine and cosine are restricted to a maximum of 1 and a minimum of -1 because these functions represent coordinates on the unit circle

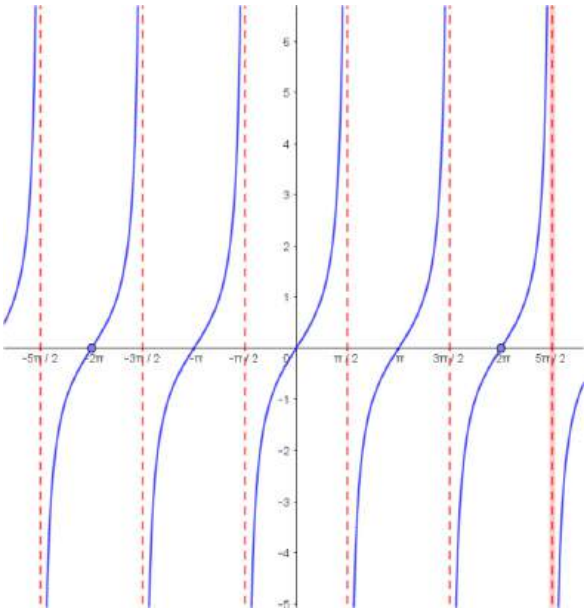
sin x



cos x



tan x



Transformations of Sine and Cosine Graphs

The graphs of sine and cosine can be shifted, stretched, or compressed. A general form for these transformations is:

$y = a\sin(bx-h)+c$ and $y = a\cos(bx-h)+c$

where

Amplitude (*a*) :
the amplitude is the peak value of the function from its midline.

Formula: amplitude = $|a|$

Period (*T*):

The period is the length of one complete cycle of the sine wave. Formula: $T = \frac{360}{|b|}$ *or* $\frac{2\pi}{|b|}$

Note: the period of tangent is 180° or π

Vertical shift (*c*):
The vertical shift moves the entire graph up or down. Formula: vertical shift = *c*

Phase shift (*h*):

The phase shift moves the graph left or right. Formula: $Phase\ shift = \frac{h}{b}$

Maximum Value (*M*):
the maximum value *M* is given by $M = c + a$

Minimum Value (*m*):
the minimum value *m* is given by $m = c - a$

The vertical shift *c* and amplitude *a* can be obtained from maximum *M* and minimum *m* using:

$c = \frac{M+m}{2}$ and $a = \frac{M-m}{2}$

Example 4:

Find the amplitude, the period, the phase shift, the vertical shift, the minimum and maximum values, Then sketch the graph of $y = \sin\left(x + 90^\circ\right)$

Solution:

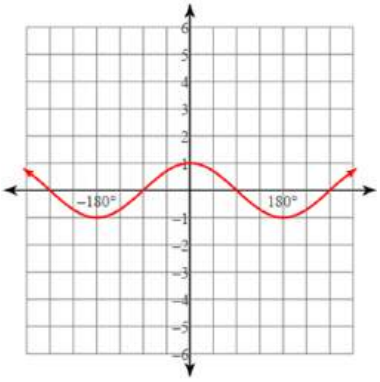
Using $y = a\sin(bx - h) + c$ we have:

a = amplitude = 1;
b = 1;

h = phase shift = $left\ 90^\circ$;

sine of any argument is always between -1 and 1 therefore:
Max= 1, Min = -1;

period = $T = \frac{360}{|b|} = \frac{360}{1} = 360^\circ$



Example 5:

Find the amplitude, the period, the phase shift, the vertical shift, the minimum and maximum values, Then sketch the graph of $y = 4\sin\left(2x - 135^\circ\right) - 1$

Solution:

Using $y = a\sin(bx - h) + c$ we have:

a = amplitude = 4;

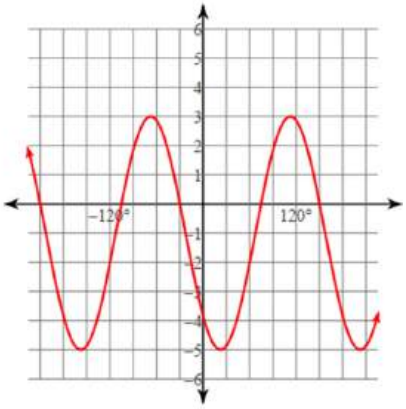
b = 2 ;

h = phase shift = $\dfrac{135^\circ}{2}$ = right 67.5° ;

sine of any argument is always between -1 and 1 therefore:

Max= 4(1) -1 = 3, Min = 4(-1) -1 = -5;

period = $T = \dfrac{360}{|b|} = \dfrac{360}{2} = 180^\circ$



Example 6:

Find the amplitude, the period, the phase shift, the minimum and maximum values, Then sketch the graph of $y = 2 + \dfrac{1}{2}\cos\left(4x + \dfrac{5}{6}\pi\right)$

Solution:

Using $y = a\cos(bx - h) + c$ we have:

a = amplitude = $\dfrac{1}{2}$;

b = 4;

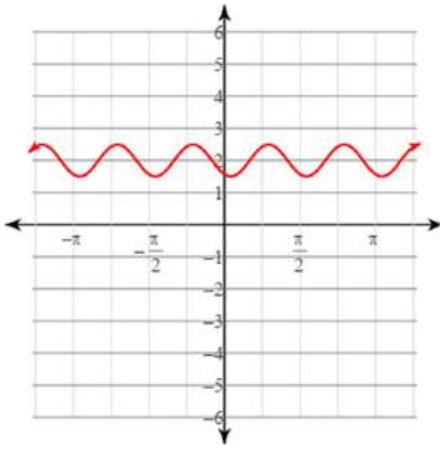
h = phase shift = $\dfrac{\frac{5}{6}\pi}{4}$ = left $\dfrac{5}{24}\pi$;

cosine of any argument is always between -1 and 1 therefore:

Max = $y = 2 + \dfrac{1}{2}\cos(1) = \dfrac{5}{2}$;

Min = $y = 2 + \dfrac{1}{2}\cos(-1) = \dfrac{3}{2}$

period = $T = \dfrac{2\pi}{|b|} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$



Example 7:

Find the amplitude, the period, the phase shift, the vertical shift, the minimum and maximum values, Then sketch the graph of $y = 4\cos\left(3x + \frac{4}{3}\pi\right) + 1$

Solution:

Using $y = a\cos(bx - h) + c$ we have:

a = amplitude = 4;

b = 3;

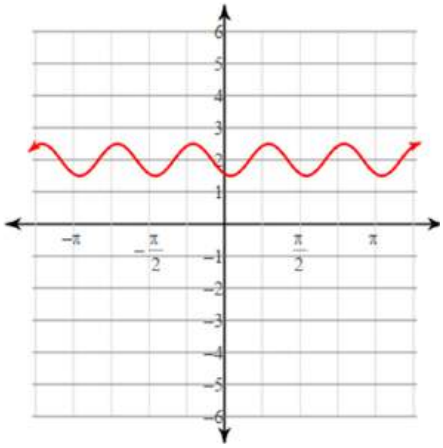
h = phase shift = $\frac{\frac{4}{3}\pi}{3}$ = left $\frac{4}{9}\pi$;

cosine of any argument is always between -1 and 1 therefore:

Max = $y = 4(1) + 1 = 5$;

Min = $y = 4(-1) + 1 = -3$

period = $T = \frac{2\pi}{|b|} = \frac{2}{3}\pi$



Miscellaneous Problems

Many business and economic metrics fluctuate in predictable, cyclical patterns. These cycles can be modeled using sine and cosine functions, allowing businesses to forecast revenue, manage inventory, and optimize pricing strategies. The key is to interpret the mathematical parameters in a business context.

Example 8:

A small boutique's monthly revenue R (in thousands of dollars) is modeled by the equation:

$R(t) = A\sin(30t) + B$

where $1 \le t \le 12$, with $t = 1$ representing January, $t = 2$ representing February, etc. The boutique's highest revenue is \$50,000 and its lowest revenue is \$20,000.

(a) Find the values of A and B.

(b) What is the boutique's revenue in March ?

(c) In what month(s) does the boutique earn exactly \$35,000?

Solution:

(a) Using $c = \frac{M + m}{2}$ and $a = \frac{M - m}{2}$ we have: $A = \frac{50 - 20}{2} = 15$ (in thousands) and $B = \frac{50 + 20}{2} = 35$ (in thousands). Therefore $R(t) = 15\sin(30t) + 35$

(b) To find the revenue in March when $t = 3$

$R(3) = 15\sin(30(3)) + 35 = 15\sin(90) + 35 = 50$(in thousand)

(c) Set $R(t) = 35 \rightarrow 35 = 15\sin(30t) + 35 \rightarrow \sin(30t) = 0$

The sine function is 0 at $0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ, \ldots$ then

$30t = 0, 180, 360, 540, 720, \ldots \rightarrow t = 0, 6, 12, 18, 24, \ldots$

With domain $1 \le t \le 12$ we have $t = 6$ = June and $t = 12$ = December

Example 9:

A warehouse's inventory level I (in hundreds of units) for a seasonal product is modeled by:

$$I(t) = A\cos(45t) + B$$

where $1 \le t \le 8$, with $t = 1$ representing January, $t = 2$ representing February, etc. The maximum inventory is 800 units and the minimum inventory is \$200 units.

(a) Find the values of A and B.

(b) What is the inventory level in February ?

(c) In what month(s) is the inventory exactly500 units?

Solution:

(a) Using $c = \dfrac{M+m}{2}$ and $a = \dfrac{M-m}{2}$ we have: $A = \dfrac{800-200}{2} = 300$ and

$B = \dfrac{800+200}{2} = 500$. Therefore $I(t) = 300(45t) + 500$

(b) in February $t = 2$

$$I(2) = 300\cos(45(2)) + 500 = 300\cos(90) + 500 = 500$$

(c) Set $I(t) = 500 \rightarrow 500 = 300\sin(45t) + 500 \rightarrow \cos(45t) = 0$

The cosine function is 0 at $90^\circ, 270^\circ, 450^\circ, 630^\circ, 810^\circ, \ldots$ then

$45t = 90, 270, 450, 630, 810, \ldots \rightarrow t = 2, 6, 10, 14, \ldots$

With domain $1 \le t \le 8$ we have $t = 2$ = February and $t = 6$ = June

Example 10:

A company's monthly revenue R (in thousands of dollars) is modeled by: $R(t) = 20\sin(45t - 135^\circ) + 50$

where $1 \le t \le 12$, with $t = 1$ representing January, $t = 2$ representing February, etc.

(a) Rewrite the function in the form $R(t) = A\sin(B(t-h)) + C$ and identify A, B, h, C.

(b) Find the amplitude, period (in months), phase shift (in months), maximum and minimum.

(c) what is the revenue in August?

(d) In what month(s) does the revenue reach its maximum?

Solution:

(a) rewrite $R(t) = 20\sin(45t - 135^\circ) + 50$ as $R(t) = 20\sin(45(t-3)) + 50$. So we have:

$$A = 20, B = 45, h = 3 \text{ months}(\text{shift to the right}), C = 50$$

(b) Amplitude = A =20, period = $\dfrac{360}{45} = 8$ months ,

$A = 20, B = 45, \textit{phase shift} = 3 \text{ months}(\text{shift to the right}), C = 50$

Max when value of sine is 1 so: $Max = 20 + 50 = 70(\textit{thousand})$

Min when value of sine is -1 so: $Min = -20 + 50 = 30(\textit{thousand})$

(c) August correspond to t = 8 so we have:

$$R(8) = 20\sin(45(8-3)) + 50 = 20\sin(225) + 50 = 20\left(\frac{\sqrt{2}}{2}\right) + 50 = 10\sqrt{2} + 50 = 35.86(\text{thousand})$$

(d) The maximum occurs when $\sin(45(t-3))=1 \rightarrow 45(t-3)=90+360n \rightarrow t-3=2+8n$

$t=5+8n$. But $1\le t\le 12$; when $n=0$ we have $t=5=May$. When $n>1$ we are outside of domain.

Example 11:

A hotel's daily revenue $R(t)$ is modeled by the sum of two sinusoidal functions

$$R(t)=200\sin(30t)+150\cos(45t)+500$$

where $1\le t\le 24$ represents the hour of the day $(t=1=1:00AM, t=2=2:00AM,...)$

a) Find the revenue at 8:00 AM
b) Find the revenue at 8:00 PM
c) Without graphing, explain why this model might better represent real hotel revenue than a single sine wave.

Solution:

$8\,AM, t=8:$

$$R(8)=200\sin(30(8))+150\cos(45(8))+500=$$

a)
$$=200\sin(240)+150\cos(360)+500=$$
$$=200\left(\frac{\sqrt{3}}{2}\right)+150(1)+500=$$
$$=200\left(-\frac{\sqrt{3}}{2}\right)+150(1)+500=$$
$$=-173.2+650=476.8\$$$

b)

$8\,PM, t=20:$

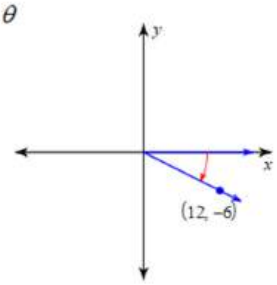
$$R(20)=200\sin(30(20))+150\cos(45(20))+500=$$
$$=200\sin(600-360)+150\cos(900-720)+500=$$
$$=200\sin(240)+150\cos(180)+500=$$
$$=200\left(-\frac{\sqrt{3}}{2}\right)+150(-1)+500=$$
$$=-173.2+350=176.8\$$$

(c) The sum of two sinusoids with different periods allows for multiple peaks/troughs per day, better capturing real-world patterns like morning and evening.

Practice Problems

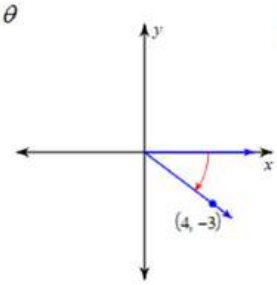
Use the given point on the terminal side of angle θ to find the value of the trigonometric function indicated.

1) $\cos\theta$



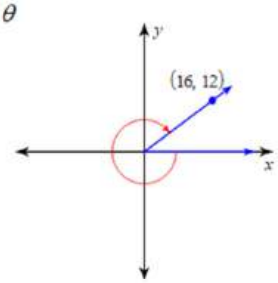
$\frac{2\sqrt{5}}{5}$

2) $\sec\theta$



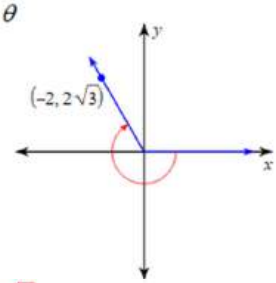
$\frac{5}{4}$

3) $\sin\theta$



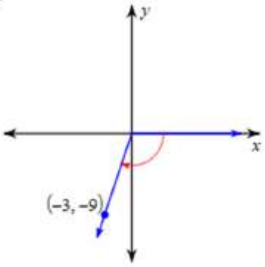
$\frac{3}{5}$

4) $\tan\theta$



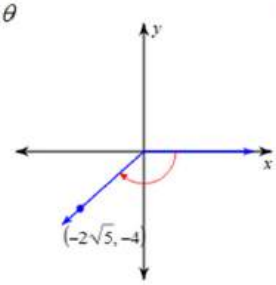
$-\sqrt{3}$

5) $\tan\theta$



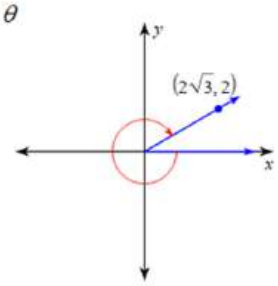
3

6) $\cos\theta$



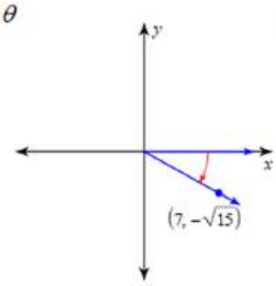
$-\frac{\sqrt{5}}{3}$

7) $\cos\theta$



$\frac{\sqrt{3}}{2}$

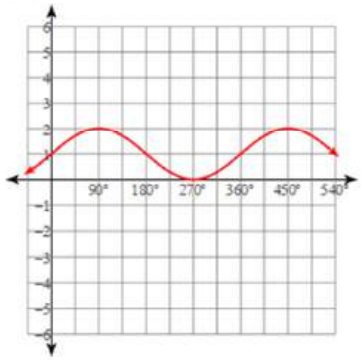
8) $\cot\theta$



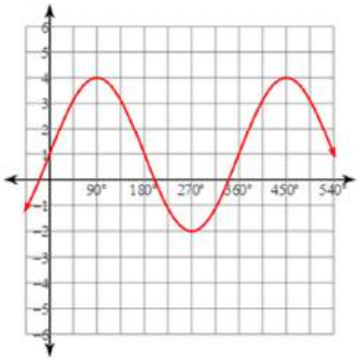
$-\frac{7\sqrt{15}}{15}$

Graph each function using degrees.

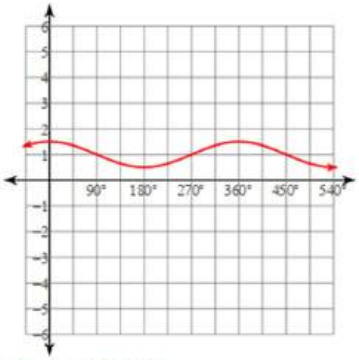
9) $y = 1 + \sin\theta$



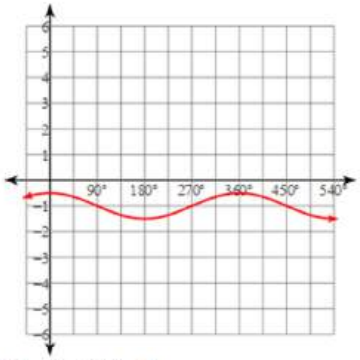
10) $y = 3\sin\theta + 1$



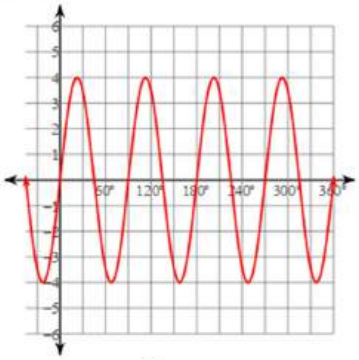
11) $y = 1 + \frac{1}{2} \cdot \cos\theta$



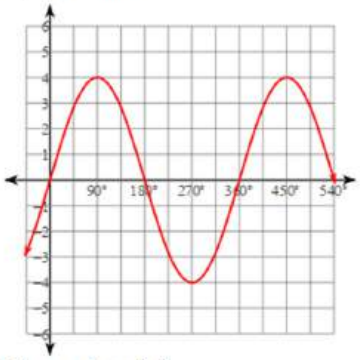
12) $y = \frac{1}{2} \cdot \cos\theta - 1$



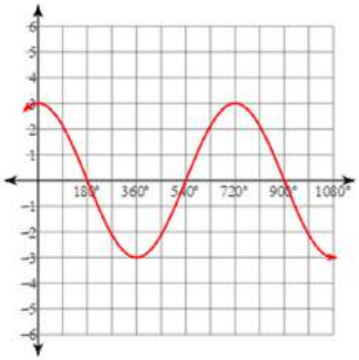
13) $y = 4\sin 4\theta$



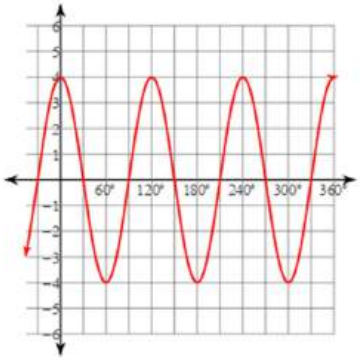
14) $y = 4\sin\theta$



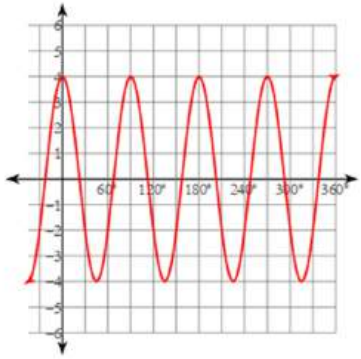
15) $y = 3\cos\frac{\theta}{2}$



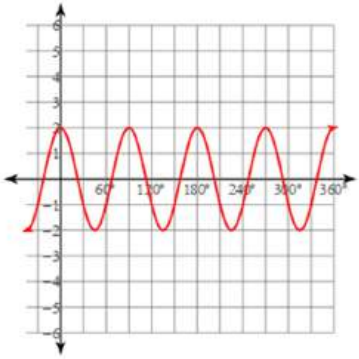
16) $y = 4\cos 3\theta$



17) $y = 4\cos 4\theta$

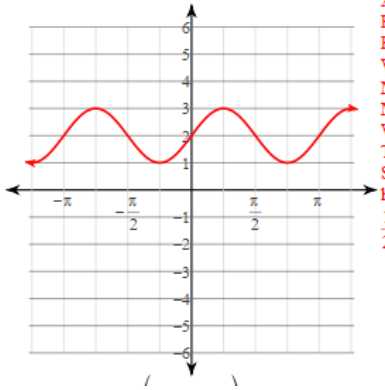


18) $y = 2\cos 4\theta$



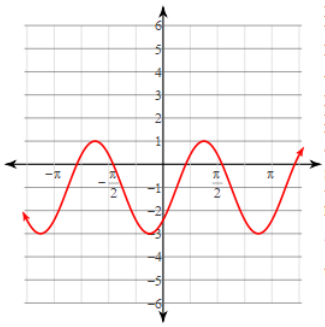
Find the amplitude, the period, the phase shift, the vertical shift, the minimum and maximum values, Then sketch the graph.

19) $y = 2 + \sin 2\theta$



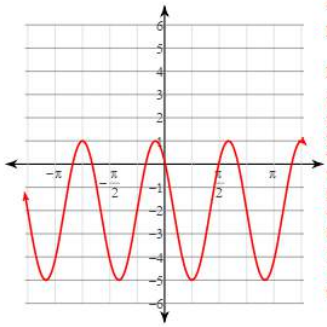
Amplitude: 1
Period: π
Phase shift: None
Vert. shift: Up 2
Min: 1
Max: 3
Vert asym: None
Transformations:
Starting with $\sin\theta$,
horizontally shrink by $\frac{1}{2}$, translate up 2

20) $y = 2\sin\left(2\theta - \frac{\pi}{4}\right) - 1$



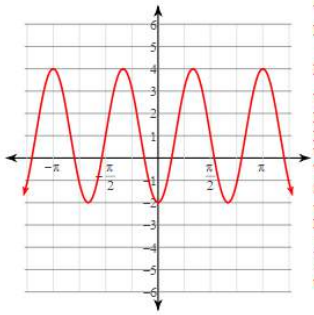
Amplitude: 2
Period: π
Phase shift: Right $\frac{\pi}{8}$
Vert. shift: Down 1
Min: -3
Max: 1
Vert asym: None
Transformations:
Starting with $\sin\theta$,
vertically stretch by 2,
horizontally shrink by $\frac{1}{2}$, translate right $\frac{\pi}{8}$,
translate down 1

21) $y = 3\sin\left(3\theta + \frac{3\pi}{4}\right) - 2$



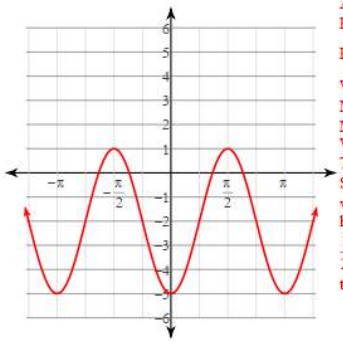
Amplitude: 3
Period: $\frac{2\pi}{3}$
Phase shift: Left $\frac{\pi}{4}$
Vert. shift: Down 2
Min: -5
Max: 1
Vert asym: None
Transformations:
Starting with $\sin\theta$,
vertically stretch by 3,
horizontally shrink by $\frac{1}{3}$, translate left $\frac{\pi}{4}$,
translate down 2

22) $y = 3\sin\left(3\theta + \frac{3\pi}{2}\right) + 1$



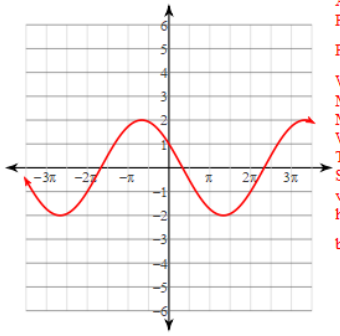
Amplitude: 3
Period: $\frac{2\pi}{3}$
Phase shift: Left $\frac{\pi}{2}$
Vert. shift: Up 1
Min: -2
Max: 4
Vert asym: None
Transformations:
Starting with $\sin\theta$,
vertically stretch by 3,
horizontally shrink by $\frac{1}{3}$, translate left $\frac{\pi}{2}$,
translate up 1

23) $y = -2 + 3\sin\left(2\theta - \frac{\pi}{2}\right)$



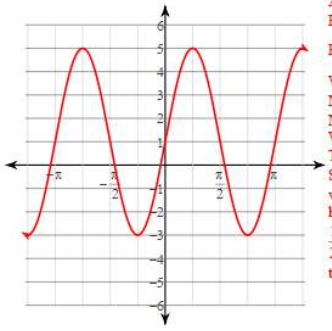
Amplitude: 3
Period: π
Phase shift: Right $\frac{\pi}{4}$
Vert. shift: Down 2
Min: -5
Max: 1
Vert asym: None
Transformations: Starting with $\sin\theta$, vertically stretch by 3, horizontally shrink by $\frac{1}{2}$, translate right $\frac{\pi}{4}$, translate down 2

24) $y = 2\cos\left(\frac{\theta}{2} + \frac{\pi}{3}\right)$



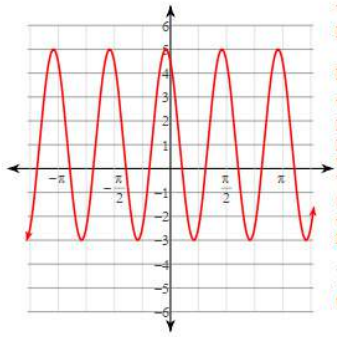
Amplitude: 2
Period: 4π
Phase shift: Left $\frac{2\pi}{3}$
Vert. shift: None
Min: -2
Max: 2
Vert asym: None
Transformations: Starting with $\cos\theta$, vertically stretch by 2, horizontally stretch by 2, translate left $\frac{2\pi}{3}$

25) $y = 1 + 4\cos\left(2\theta - \frac{\pi}{2}\right)$



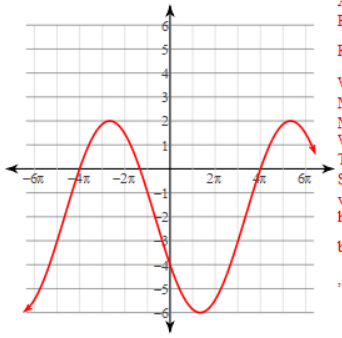
Amplitude: 4
Period: π
Phase shift: Right $\frac{\pi}{4}$
Vert. shift: Up 1
Min: -3
Max: 5
Vert asym: None
Transformations: Starting with $\cos\theta$, vertically stretch by 4, horizontally shrink by $\frac{1}{2}$, translate right $\frac{\pi}{4}$, translate up 1

26) $y = 4\cos\left(4\theta - \frac{11\pi}{6}\right) + 1$



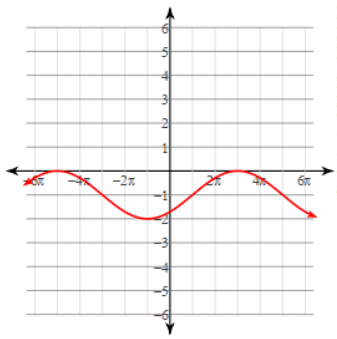
Amplitude: 4
Period: $\frac{\pi}{2}$
Phase shift: Right $\frac{11\pi}{24}$
Vert. shift: Up 1
Min: -3
Max: 5
Vert asym: None
Transformations: Starting with $\cos\theta$, vertically stretch by 4, horizontally shrink by $\frac{1}{4}$, translate right $\frac{11\pi}{24}$, translate up 1

27) $y = 4\cos\left(\frac{\theta}{4} + \frac{2\pi}{3}\right) - 2$



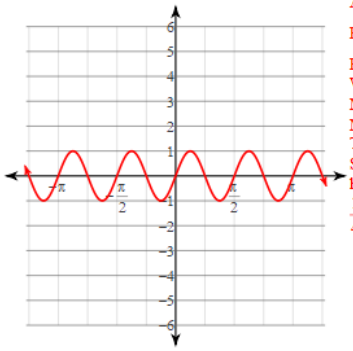
Amplitude: 4
Period: 8π
Phase shift: Left $\frac{8\pi}{3}$
Vert. shift: Down 2
Min: -6
Max: 2
Vert asym: None
Transformations: Starting with $\cos\theta$, vertically stretch by 4, horizontally stretch by 4, translate left $\frac{8\pi}{3}$, translate down 2

28) $y = \cos\left(\frac{\theta}{4} - \frac{3\pi}{4}\right) - 1$



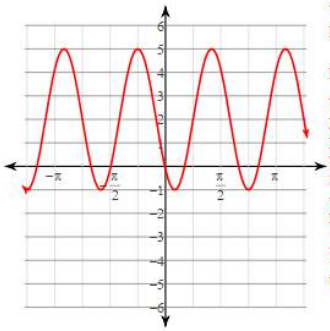
Amplitude: 1
Period: 8π
Phase shift: Right 3π
Vert. shift: Down 1
Min: -2
Max: 0
Vert asym: None
Transformations: Starting with $\cos\theta$, horizontally stretch by 4, translate right 3π, translate down 1

29) $y = \sin 4\theta$



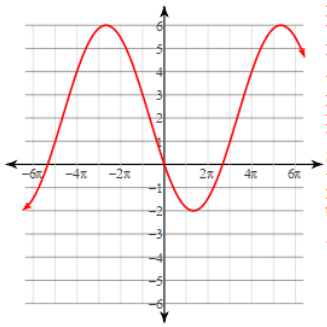
Amplitude: 1
Period: $\frac{\pi}{2}$
Phase shift: None
Vert. shift: None
Min: -1
Max: 1
Transformations: Starting with $\sin\theta$, horizontally shrink by $\frac{1}{4}$

30) $y = 2 + 3\sin\left(3\theta - \frac{3\pi}{4}\right)$



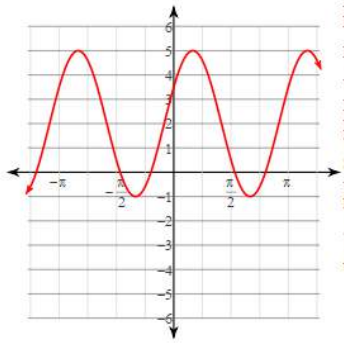
Amplitude: 3
Period: $\frac{2\pi}{3}$
Phase shift: Right $\frac{\pi}{4}$
Vert. shift: Up 2
Min: -1
Max: 5
Transformations: Starting with $\sin\theta$, vertically stretch by 3, horizontally shrink by $\frac{1}{3}$, translate right $\frac{\pi}{4}$, translate up 2

31) $y = 4\sin\left(\frac{\theta}{4} - \frac{5\pi}{6}\right) + 2$



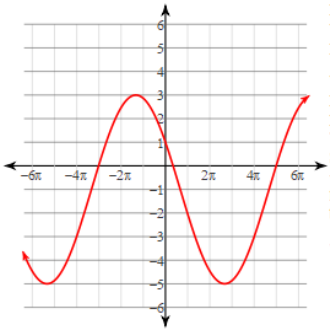
Amplitude: 4
Period: 8π
Phase shift: Right $\frac{10\pi}{3}$
Vert. shift: Up 2
Min: -2
Max: 6
Transformations: Starting with $\sin\theta$, vertically stretch by 4, horizontally stretch by 4, translate right $\frac{10\pi}{3}$, translate up 2

32) $y = 3\sin\left(2\theta + \frac{\pi}{6}\right) + 2$



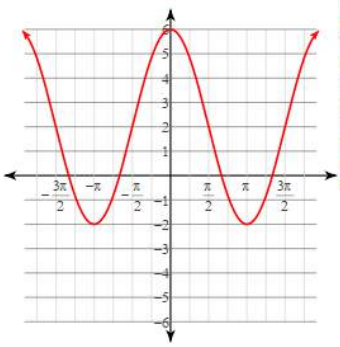
Amplitude: 3
Period: π
Phase shift: Left $\frac{\pi}{12}$
Vert. shift: Up 2
Min: -1
Max: 5
Transformations: Starting with $\sin\theta$, vertically stretch by 3, horizontally shrink by $\frac{1}{2}$, translate left $\frac{\pi}{12}$, translate up 2

33) $y = -1 + 4\sin\left(\frac{\theta}{4} + \frac{5\pi}{6}\right)$



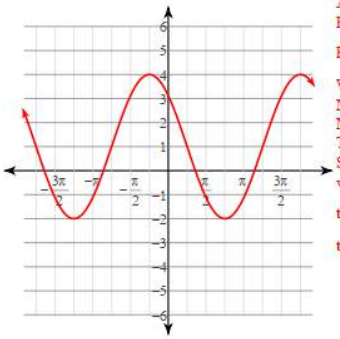
Amplitude: 4
Period: 8π
Phase shift: Left $\frac{10\pi}{3}$
Vert. shift: Down 1
Min: -5
Max: 3
Transformations: Starting with $\sin\theta$, vertically stretch by 4, horizontally stretch by 4, translate left $\frac{10\pi}{3}$, translate down 1

34) $y = 4\cos\theta + 2$



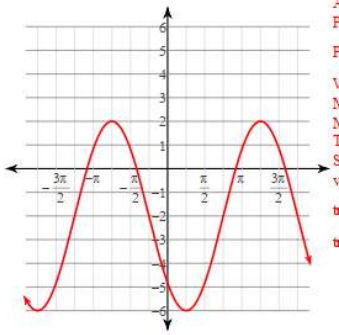
Amplitude: 4
Period: 2π
Phase shift: None
Vert. shift: Up 2
Min: -2
Max: 6
Transformations: Starting with $\cos\theta$, vertically stretch by 4, translate up 2

35) $y = 3\cos\left(\theta + \frac{\pi}{4}\right) + 1$



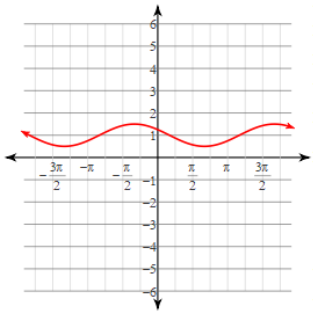
Amplitude: 3
Period: 2π
Phase shift: Left $\frac{\pi}{4}$
Vert. shift: Up 1
Min: -2
Max: 4
Transformations:
Starting with cos θ,
vertically stretch by 3,
translate left $\frac{\pi}{4}$,
translate up 1

36) $y = 4\cos\left(\theta - \frac{5\pi}{4}\right) - 2$



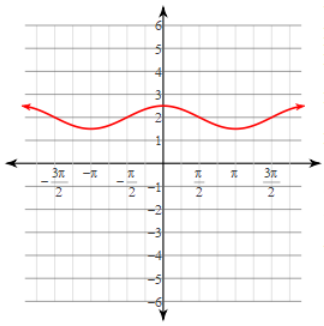
Amplitude: 4
Period: 2π
Phase shift: Right $\frac{5\pi}{4}$
Vert. shift: Down 2
Min: -6
Max: 2
Transformations:
Starting with cos θ,
vertically stretch by 4,
translate right $\frac{5\pi}{4}$,
translate down 2

37) $y = \frac{1}{2} \cdot \cos\left(\theta + \frac{\pi}{3}\right) + 1$



Amplitude: $\frac{1}{2}$
Period: 2π
Phase shift: Left $\frac{\pi}{3}$
Vert. shift: Up 1
Min: $\frac{1}{2}$
Max: $\frac{3}{2}$
Transformations:
Starting with cos θ,
vertically shrink by $\frac{1}{2}$,
translate left $\frac{\pi}{3}$,
translate up 1

38) $y = \frac{1}{2} \cdot \cos\theta + 2$



Amplitude: $\frac{1}{2}$
Period: 2π
Phase shift: None
Vert. shift: Up 2
Min: $\frac{3}{2}$
Max: $\frac{5}{2}$
Transformations:
Starting with cos θ,
vertically shrink by $\frac{1}{2}$,
translate up 2

39) The depth of the water D at the end of a pier varies with the tides throughout the day and it is modeled by the equation $D = A\cos(30t) + b$ where $0 \le t \le 24$.

(i) Find *A* and *b* — *A=4, b=6*

(ii) Find the depth of the water at 10 A.M. — *8 meters*

(iii) At what time, during the day is the depth 6 meters? — *at 3 AM, 9AM, 3PM and 9PM*

40) The approximate number of day light N between sunrise and sunset varies throughout the year and it is modeled by the equation $N = A\sin(d) + B$ where $1 \le d \le 365$ with a maximum of 16 hours and a minimum of 10 hours.

(i) find A, and B — A=3, B=13

(ii) Find the number of hours on January 30? — 14.5 hours

41) The price P (in dollars) of a stock is modeled by: $P = A\cos(45t) + B$ where $0 \le t \le 24$.

(i) Find A and B if the highest price is \$120 and the lowest is \$80. *A=20, B= 100*
(ii) Find the price at 9 A.M. *\$114.14*
(iii) At what time(s) is the price \$100? *At 2 A.M. and 6A.M.*

42) A company's daily revenue R (in thousands of dollars) fluctuates due to seasonal demand and is modeled by: $R = A\sin(30t) + B$ where $0 \le t \le 24$.

(i) Find A and B if the highest revenue is \$50,000 and the lowest is \$10,000. *A=20, B= 30*
(ii) Find the revenue at 12 P.M. (noon). *\$30,000*
(iii) At what time(s) during the day is the revenue \$30,000? *At midnight, 6A.M.Noon, 6P.M.*

43) A warehouse tracks inventory $I(t)$ (in units) over a 2-year period: $I(t) = 150\cos\left(\frac{\pi}{12}t - \frac{\pi}{3}\right) + 400$

where $1 \le t \le 24$ represents months ($t = 1$ = January of Year 1).

(a) Rewrite in the form $I(t) = A\cos(B(t-h)) + C$ (use radians).

(b) Find the amplitude, period (in months), phase shift (in months), max, and min.
(c) What is the inventory in October of Year 2?
(d) In what month(s) does the inventory hit its minimum?

(a) $I(t) = 150\cos\left(\frac{\pi}{12}(t-4)\right) + 400$; $A = 150; B = \frac{\pi}{12}; h = 4; C = 400$

(b) Amplit = 150; period = $B = \frac{2\pi}{\frac{\pi}{12}} = 24$ months ; shift = 4 months to the right; Max=550; min = 250

(c) 400 units
(d) April of year 2

44) A company's monthly profit $P(t)$ (in thousands of dollars) is modeled by

$P(t) = 12\sin(60t + 240) + 18$

where $1 \le t \le 18$ represents months ($t = 1$ = January).

(a) Rewrite in the form $P(t) = A\sin(B(t-h)) + C$ and identify all parameters.

(b) How many full cycles occur in the 18-month period?
(c) What is the profit in July?
(d) In what month(s) does the profit equal exactly \$30,000?

(a) $P(t) = 12\sin(60B(t-4)) + 18$; $A = 12; B = 60; h = -4; C = 18$

(b) period = $\frac{360}{60} = 6$ months ; number of cycles in 18 months: $\frac{18}{6} = 3$ full cycles

(c) $P(7) \approx 7608$

(d) $t = -2.5 + 6k$

$k = 1$; $t = 3.5$ between March and April

$k = 2$; $t = 9.5$ between September and October

$k = 3$; $t = 15.5$ between March and April of the next year

45) A seasonal product's daily sales $S(t)$ (in units) depend on the average daily temperature, which itself follows a yearly cycle. The model is $S(t) = 50\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 100$

where $1 \le t \le 12$ represents months ($t = 1$ = January). However, the company runs a 30-day promotion in the middle of the year that increases sales by a constant 20 units per day for the entire month of July.

(a) Find the baseline sales (without the promotion) in July.

(b) Find the actual sales (with the promotion) in July.

(c) In what month(s) does the baseline sales equal exactly 100 units?

(a) $S(7) \approx 143.3$ units

(b) $S(7)20 \approx 163$ units

(c) $t = 3 + 6k$

$k = 0;\ t = 3$ March

$k = 1;\ t = 9$ September

3.3 Trigonometric Identities and Analytic Evaluation

Topics: addition/subtraction, co-function and even-odd identities, double angle formulas

Review

1) The temperature T(in °C) of a lake during one day is modeled by $T(t) = A\sin(45t) + B$, $0 \le t \le 24$ where $t$is measured in hours.The highest temperature is 26°C and the lowest is 14°C.

a) Find $A$and B. A = 6, B = 20; $T(t) = 6\sin(45t) + 20$

b) Find the temperature at 6 P.M. 26°

c) During what times is the temperature exactly 20°C? $t = 0, 4, 8, 12, 16, 20, 24$

2) The number of tourists (in thousands) is modeled by $N(t) = 35\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 120$

Where $0 \le t \le 24$.

a) Rewrite the function in the form $N(t) = A\sin(B(h-t)) + C$ $N(t) = 35\sin\left(\frac{\pi}{6}(t-3)\right) + 120$

b) Determine the amplitude, period, phase shift, maximum, and minimum.

A=35, period = 12 months shift = 3 months right; max = 155; min = 85

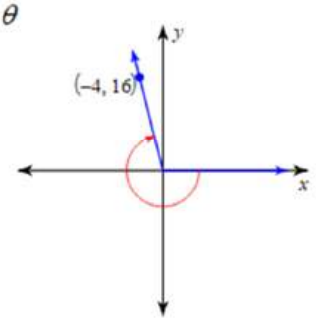
c) Estimate the number of tourists in August. $N(8) = 137.5$(thousand tourists)

d) During which month(s) are there exactly 120 thousand tourists?

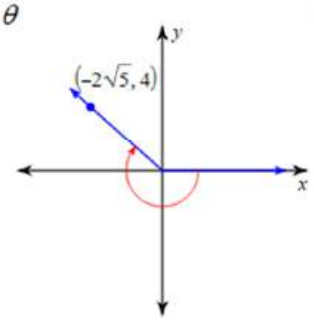
$t = 3 + 6k$; so 3 and 9; March and September

Use the given point on the terminal side of angle θ to find the value of the trigonometric function indicated.

3) csc θ $\frac{\sqrt{17}}{4}$



4) sec θ $-\frac{3\sqrt{5}}{5}$



Addition and Subtraction Formulas

Many trig expressions involve sums or differences of angles that can't be directly evaluated from the unit circle. We use addition and subtraction formulas to rewrite these expressions using simpler, known angles. These formulas are foundational—they are essential for deriving later identities like double-angle, half-angle, and product-to-sum formulas.

For any angles A and B,

Sine

$$\sin(A+B)=\sin A\cos B+\sin B\cos A$$

$$\sin(A-B)=\sin A\cos B-\sin B\cos A$$

the sign in the result matches the sign between the angles.

Cosine

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

Unlike the sine formulas, the cosine formulas have the **opposite sign**.

Tangent

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

These formulas should be used only when the denominator is not zero

Example 1

Evaluate $\sin 75^\circ$

Solution:

Write $75^\circ=45^\circ+30^\circ$

Using $\sin(A+B)=\sin A\cos B+\sin B\cos A$ we have:

$$\sin(75^\circ)=\sin 45^\circ\cos 30^\circ+\sin 30^\circ\cos 45^\circ=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$$

Example 2

Evaluate $\cos 15^\circ$

Solution:

Write $15^\circ=45^\circ-30^\circ$

Using $\cos(A-B)=\cos A\cos B+\sin A\sin B$ we have:

$$\cos(15^\circ)=\cos 45^\circ\cos 30^\circ+\sin 30^\circ\sin 45^\circ=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$$

Example 3

Let x and y be angles such that: $\sin x=-\frac{5}{13},\pi<x<\frac{3}{2}\pi$ and $\cos y=-\frac{3}{5},\frac{\pi}{2}<y<\pi$.

Find the exact value of $\tan(x-y)$

Solution:

To find $\tan(x-y)$ we need to find first both $\tan x$ and $\tan y$.

$$\cos x = -\sqrt{1-\left(\frac{5}{13}\right)^2} = -\frac{12}{13} \rightarrow \tan x = \frac{\sin x}{\cos x} = \frac{-\frac{5}{13}}{-\frac{12}{13}} = \frac{5}{12}$$

$$\sin y = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5} \rightarrow \tan y = \frac{\sin y}{\cos y} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

Now: $\tan(x-y) = \dfrac{\tan x - \tan y}{1+\tan x \tan y} = \dfrac{\frac{5}{12} - \left(-\frac{4}{3}\right)}{1+\left(\frac{5}{12}\right)\left(-\frac{4}{3}\right)} = \dfrac{\frac{21}{12}}{\frac{4}{9}} = \dfrac{63}{16}$

Example 4

Find the exact value of: $\sin 10^\circ \cos 20^\circ + \sin 15^\circ \cos 30^\circ + \cos 10^\circ \sin 20^\circ + \cos 15^\circ \sin 30^\circ$

Solution:

Rewrite as:

$(\sin 10^\circ \cos 20^\circ + \cos 10^\circ \sin 20^\circ) + (\sin 15^\circ \cos 30^\circ + \cos 15^\circ \sin 30^\circ)$

Apply $\sin(A+B) = \sin A \cos B + \sin B \cos A$ to have:

$(\sin 10^\circ \cos 20^\circ + \cos 10^\circ \sin 20^\circ) = \sin(10^\circ + 20^\circ) = \sin 30^\circ$ and

$(\sin 15^\circ \cos 30^\circ + \cos 15^\circ \sin 30^\circ) = \sin(15^\circ + 30^\circ) = \sin 45^\circ$. Therefore:

$$\sin 10^\circ \cos 20^\circ + \sin 15^\circ \cos 30^\circ + \cos 10^\circ \sin 20^\circ + \cos 15^\circ \sin 30^\circ = \sin 30^\circ + \sin 45^\circ = \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{3}+\sqrt{2}}{2}$$

Co-function Identities

The co-function identities relate complementary angles.

1. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$;
2. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$;
3. $\tan\left(\frac{\pi}{2} - x\right) = \cot x$;
4. $\cot\left(\frac{\pi}{2} - x\right) = \tan x$;
5. $\sec\left(\frac{\pi}{2} - x\right) = \csc x$;
6. $\csc\left(\frac{\pi}{2} - x\right) = \sec x$

Example 5

Simplify $\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)$

Solution:

Since $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ we have: $\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right) = \cos x + \sin x$

Example 6

Show that $\dfrac{\sin(90^\circ - x)}{\cos(90^\circ - x)} + \dfrac{\cos(90^\circ - x)}{\sin(90^\circ - x)} = \sec x \csc x$

Solution:

Since $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$ and $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$ we have:

$$\frac{\sin(90^\circ - x)}{\cos(90^\circ - x)} + \frac{\cos(90^\circ - x)}{\sin(90^\circ - x)} = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \left(\frac{1}{\sin x}\right)\left(\frac{1}{\cos x}\right) = \sec x \csc x$$

Example 7

Given that $\sin(90^\circ - x) = \dfrac{5}{13}; 0 < x < 90^\circ$, evaluate $\dfrac{\tan(90^\circ - x) - \sin x}{\cos x}$

Solution:

$\sin(90^\circ - x) = \dfrac{5}{13} = \cos x$. It follows that $\sin x = \sqrt{1 - \left(\dfrac{5}{13}\right)^2} = \dfrac{12}{13}$. Then $\sin x = \cos(90^\circ - x) = \dfrac{12}{13}$

Therefore $\tan(90^\circ - x) = \dfrac{\sin(90^\circ - x)}{\cos(90^\circ - x)} = \dfrac{\frac{5}{13}}{\frac{12}{13}} = \dfrac{5}{12}$

Substitute: $\dfrac{\tan(90^\circ - x) - \sin x}{\cos x} = \dfrac{\frac{5}{12} - \frac{12}{13}}{\frac{5}{13}} = \dfrac{-\frac{79}{156}}{\frac{5}{13}} = -\dfrac{79}{60}$

Even-Odd Functions

Negative angles can be simplified using the parity of each trigonometric function.

Even Functions: $\cos(-x) = \cos x$ and $\sec(-x) = \sec x$

Odd Functions: $\sin(-x) = -\sin x$; $\tan(-x) = -\tan x$; $\cot(-x) = -\cot x$; $\csc(-x) = -\csc x$

Example 8

Simplify $\dfrac{\sin(-x)\cos(-x)}{\tan(-x)}$

Solution:

$$\frac{\sin(-x)\cos(-x)}{\tan(-x)} = \frac{-\sin x \cos x}{-\tan x} = \frac{\sin x \cos x}{\frac{\sin x}{\cos x}} = (\sin x \cos x)\left(\frac{\cos x}{\sin x}\right) = \cos^2 x$$

Example 9

Given that $\sin x = \frac{8}{17}; 0 < x < 90^\circ$, evaluate $\frac{\sin(-x)+\tan(-x)}{\cos(-x)}$

Solution:

First simplify: $\frac{\sin(-x)+\tan(-x)}{\cos(-x)} = \frac{-.\sin(x)-\tan(x)}{\cos(x)}$

Since $\sin x = \frac{8}{17}$ we have $\cos x = \sqrt{1-\left(\frac{8}{17}\right)^2} = \frac{15}{17}$; it follows that $\tan x = \frac{\sin x}{\cos x} = \frac{\frac{8}{17}}{\frac{15}{17}} = \frac{8}{15}$

Therefore: $\frac{\sin(-x)+\tan(-x)}{\cos(-x)} = \frac{-.\sin(x)-\tan(x)}{\cos(x)} = \frac{-.\frac{8}{17}-\frac{8}{15}}{\frac{15}{17}} = -\frac{256}{225}$

Double Angle Formulas

The addition formulas immediately produce the double-angle identities.

Sine

$\sin 2x = 2\sin x \cos x$

Cosine

Three equivalent forms are commonly used.

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$

Tangent

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

Example 10

Simplify $1 - 2\sin^2 x + \cos 2x$

Solution:

$1 - 2\sin^2 x + \cos 2x =$

$= \cos 2x + \cos 2x =$

$= 2\cos 2x$

Example 11

Simplify $\frac{1-\cos 2x}{\sin 2x}$

Solution:

$\frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{\sin 2x} = \frac{2\sin^2 x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x$

Example 12

Simplify $\dfrac{\cos 2x - 1 + \sin 2x}{\sin x}$

Solution:

$$\frac{\cos 2x - 1 + \sin 2x}{\sin x} == \frac{\left(-2\sin^2 x\right) + \left(2\sin x \cos x\right)}{\sin x} = \frac{2\sin x\left(\cos x - \sin x\right)}{\sin x} = 2\left(\cos x - \sin x\right)$$

Miscellaneous Problems

Strategy for Solving Trigonometric Identity Problems

When working with identities, ask the following questions:

Can I rewrite everything in terms of sine and cosine?

Can I use a reciprocal or quotient identity?

Does a Pythagorean identity apply?

Can I use the conjugate?

Would factoring simplify the expression?

Can an addition, subtraction, co-function, or double-angle identity be used?

Following this strategy will simplify most trigonometric expressions

Example 13

Simplify $\dfrac{\sin x}{1 + \cos x}$

Solution:

Multiply numerator and denominator by the conjugate.

$$\frac{\sin x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} = \frac{\sin x\left(1-\cos x\right)}{1-\cos^2 x}$$

Using $1 - \cos^2 x = \sin^2 x$

$$\frac{\sin x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x} = \frac{\sin x\left(1-\cos x\right)}{1-\cos^2 x} = \frac{\sin x\left(1-\cos x\right)}{\sin^2 x} = \frac{1-\cos x}{\sin x} = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \csc x - \cot x$$

Example 14

Evaluate $2\cos^2 45^\circ + 4\tan 60^\circ \sin 60^\circ - 2\cot^2 30^\circ + \csc 30^\circ$

Solution:

$$2\cos^2 45^\circ + 4\tan 60^\circ \sin 60^\circ - 2\cot^2 30^\circ + \csc 30^\circ =$$

$$= 2\left(\frac{\sqrt{2}}{2}\right)^2 + 4\left(\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\cos 30^\circ}{\sin 30^\circ}\right)^2 + \frac{1}{\sin 30^\circ} =$$

$$= 1 + 6 - 2\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)^2 + \frac{1}{\frac{1}{2}} =$$

$$= 1 + 6 - 2(3) + 2 =$$

$$= 2 + 1 =$$

$$= 3$$

Example 15

Evaluate $3\sin\frac{\pi}{2}-2\cos\pi+5\tan^2\frac{\pi}{4}-4\sec^2\frac{\pi}{6}\sin^2\frac{\pi}{3}$

Solution:

$$3\sin\frac{\pi}{2}-2\cos\pi+5\tan^2\frac{\pi}{4}-4\sec^2\frac{\pi}{6}\sin^2\frac{\pi}{3}$$

$$=3(1)-2(-1)+5(1)^2-4\left(\frac{1}{\cos\frac{\pi}{6}}\right)^2\left(\frac{\sqrt{3}}{2}\right)^2=$$

$$=3+2+5-4\left(\frac{1}{\frac{\sqrt{3}}{2}}\right)^2\left(\frac{3}{4}\right)=$$

$$=10-4\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)=$$

$$=6$$

Example 16

Evaluate $2\sin150^\circ-3\cos240^\circ+\tan315^\circ-\sqrt{2}\sec225^\circ$

Solution:

$$2\sin150^\circ-3\cos240^\circ+\tan315^\circ-\sqrt{2}\sec225^\circ=$$

$$=2\sin(180^\circ-30^\circ)-3\cos(180^\circ+60^\circ)+\tan(360^\circ-45^\circ)-\sqrt{2}\sec(180^\circ+45^\circ)=$$

$$=2\sin(30^\circ)-3(-\cos60^\circ)-\tan(45^\circ)-\sqrt{2}(-\sec45^\circ)=$$

$$=2\left(\frac{1}{2}\right)-3\left(-\frac{1}{2}\right)-(1)-\sqrt{2}\left(-\frac{2}{\sqrt{2}}\right)=$$

$$=1+\frac{3}{2}-1+2=$$

$$=\frac{7}{2}$$

Example 17

Simplify $\dfrac{\sin(-x)\cos\left(\frac{\pi}{2}-x\right)+\cos(-x)\sin\left(\frac{\pi}{2}-x\right)}{\cos2x}$

Solution:

$$\frac{\sin(-x)\cos\left(\frac{\pi}{2}-x\right)+\cos(-x)\sin\left(\frac{\pi}{2}-x\right)}{\cos2x}=$$

$$=\frac{-\sin x\sin x+\cos x\cos x}{\cos^2x-\sin^2x}=\frac{-\sin^2x+\cos^2x}{\cos^2x-\sin^2x}=$$

$$=\frac{\cos^2x-\sin^2x}{\cos^2x-\sin^2x}=\frac{\cos2x}{\cos2x}=1$$

Example 18

Simplify $\frac{\sin 2x}{1+\cos 2x}\cdot\tan\left(\frac{\pi}{2}-x\right)-\cos^2(-x)$

Solution:

$$\frac{\sin 2x}{1+\cos 2x}\cdot\tan\left(\frac{\pi}{2}-x\right)-\cos^2(-x)=$$

$$=\frac{2\sin x\cos x}{1+\left(2\cos^2 x-1\right)}\cdot\cot x-\cos^2 x=$$

$$=\frac{2\sin x\cos x}{2\cos^2 x}\cdot\frac{\cos x}{\sin x}-\cos^2 x=$$

$$=1-\cos^2 x=$$

$$=\sin^2 x$$

Example 19

Simplify $\frac{4\sin(-x)\cos(-x)\cos(2x)}{\sin\left(\frac{\pi}{2}-4x\right)}$

Solution:

$$\frac{4\sin(-x)\cos(-x)\cos(2x)}{\sin\left(\frac{\pi}{2}-4x\right)}=\frac{-4\sin x\cos x\cos(2x)}{\cos 4x}=$$

$$=\frac{-2(2\sin x\cos x)\cos(2x)}{\cos 4x}=\frac{-2\sin(2x)\cos(2x)}{\cos 4x}=$$

$$=\frac{-\sin(4x)}{\cos 4x}=-\tan(4x)$$

Example 20

Show $\frac{\cos x}{1-\sin x}=\sec x+\tan x$

Solution:

$$\frac{\cos x}{1-\sin x}\cdot\frac{1+\sin}{1+\sin x}=\frac{\cos x(1+\sin x)}{1-\sin^2 x}=$$

$$=\frac{\cos x(1+\sin x)}{\cos^2 x}=\frac{(1+\sin x)}{\cos x}=$$

$$=\frac{1}{\cos x}+\frac{\sin x}{\cos x}=\sec x+\tan x$$

Example 21

Simplify $\dfrac{\sin x+\tan x}{\tan x-\sin x}$

Solution:

$$\frac{\sin x+\tan x}{\tan x-\sin x}=\frac{\sin x+\dfrac{\sin x}{\cos x}}{\dfrac{\sin x}{\cos x}-\sin x}=\frac{\dfrac{\sin x\cos x+\sin x}{\cos x}}{\dfrac{\sin x-\sin x\cos x}{\cos x}}=$$

$$=\frac{\sin x\cos x+\sin x}{\sin x-\sin x\cos x}=\frac{\sin x(\cos x+1)}{\sin x(1-\cos x)}=$$

$$=\frac{1+\cos x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}=\frac{1+\cos^2 x}{1-\cos^2 x}=$$

$$=\frac{1+\cos^2 x}{\sin^2 x}=\frac{1}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}=\csc^2 x+\cot^2 x$$

Example 22

Simplify $\dfrac{\sin x+\cos x-1}{\sin x-\cos x+1}$

Solution:

$$\frac{\sin x+\cos x-1}{\sin x-\cos x+1}=\frac{\sin x+(\cos x-1)}{\sin x-(\cos x-1)}=$$

$$=\frac{\sin x+(\cos x-1)}{\sin x-(\cos x-1)}\cdot\frac{\sin x+(\cos x-1)}{\sin x+(\cos x-1)}=$$

$$=\frac{\left[\sin x+(\cos x-1)\right]^2}{\sin^2 x-\left[(\cos x-1)\right]^2}=\frac{\sin^2 x+2\sin x(\cos x-1)+(\cos x-1)^2}{\sin^2 x-(\cos^2 x-2\cos x+1)}=$$

$$=\frac{\sin^2 x+2\sin x\cos x-2\sin x+\cos^2 x-2\cos x+1}{\sin^2 x-\cos^2 x+2\cos x-1}=$$

$$=\frac{(\sin^2 x+\cos^2 x)+2\sin x\cos x-2\sin x-2\cos x+1}{(\sin^2 x-1)-\cos^2 x+2\cos x}=\frac{1+2\sin x\cos x-2\sin x-2\cos x+1}{-\cos^2 x-\cos^2 x+2\cos x}=$$

$$=\frac{2+2\sin x\cos x-2\sin x-2\cos x}{-2\cos^2 x+2\cos x}=\frac{1+\sin x\cos x-\sin x-\cos x}{-\cos^2 x+\cos x}=\frac{\sin x\cos x-\sin x-\cos x+1}{\cos x(1-\cos x)}=$$

$$=\frac{\sin x(\cos x-1)-1(\cos x-1)}{\cos x(1-\cos x)}=\frac{(\sin x-1)(\cos x-1)}{-\cos x(\cos x-1)}=$$

$$=\frac{(\sin x-1)}{-\cos x}=\frac{\sin x}{-\cos x}+\frac{1}{\cos x}=\sec x-\tan x$$

Practice Problems

Evaluate the following

1) $4\sin 30 - \sec 60 + \sqrt{2}\csc 45 + \cos 90 - 3\sec 0 + \cot 45$

0

2) $4\cos 0 - 2\sec\frac{\pi}{3} + 2\csc\frac{\pi}{4} - 4\sin\frac{\pi}{4} + \cot\frac{\pi}{2}$

0

3) $\tan 0 + 4\cos 30\sin 60 - \sqrt{2}\cos 45 - 6\sin 90$

-4

4) $\cos 0 + \sin 90 - 3\cos 180 + 5\sin^2 270 - \sin 180 + 7\cos 270$

10

5) $\cot\frac{\pi}{2} - 3\sec\frac{\pi}{4} + \csc\frac{\pi}{6}\sec\frac{\pi}{6} - 8\cot\frac{\pi}{3}\cos\frac{\pi}{3}$

6) Express $\sin x$ in terms of $\cos x$

$\sin x = \sqrt{1 - \cos^2 x}$

7) Express $\sec x$ in terms of $\tan x$

$\sec x = \sqrt{1 + \tan^2 x}$

8) Express $\sin x \sec x$ as $\tan x$

$\tan x = \sin x \sec x$

Using the Fundamental Trigonometry Relations, Simplify the following

9) $\frac{\sin^3 x - \sin x}{\cos x} + 2\tan x + \frac{\sin^2 x}{\tan x}$

$2\tan x$

10) $\frac{1}{2} \cdot \cos x + \frac{\tan^2 x}{1 + \tan x} - \sin^2 x + \frac{1}{2} \cdot \frac{\sin^2 x}{\cos x \tan^2 x}$

$\cos x$

11) $\frac{\cos^2 x}{1 - \cos^2 x} - \tan x + \frac{1 - \sin^2 x}{\cos^2 x} - \frac{1}{\sin^2 x}$

$-\tan x$

12) $\cot(75°)$

$2 - \sqrt{3}$

13) $\sin x \tan x + \cos x \cdot (1 - \sin x) + \tan x \cos^2 x$

$\sec x$

14) $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x}$

$2\csc x$

15) $\left(\frac{1}{\cos^2 x} + \tan^2 x\right) \cdot \frac{1}{\sec^2 x}$

$1 + \sin^2 x$

16) $(\sin x + \cos x)^2 - 2\tan x \cos^2 x + 2\sin^2 x - 1$

$2\sin^2 x$

Evaluate the following

17) $\sin(165°)$ $\frac{\sqrt{6}-\sqrt{2}}{4}$

18) $\cos\frac{7}{12}\pi$ $\frac{\sqrt{2}-\sqrt{6}}{4}$

19) $\tan(195°)$
$2-\sqrt{3}$

20) $\sin\frac{17}{12}\pi$ $\frac{-\sqrt{6}-\sqrt{2}}{4}$

21) $\sec\frac{17}{12}\pi$
$-\sqrt{6}-\sqrt{2}$

22) $(\sec^2 x+\csc^2 x)\cdot\cot^2 x$
$\csc^4 x$

Simplify the following

23) $\sin\left(\frac{\pi}{6}+x\right)$ $\frac{1}{2}\cdot\cos x+\frac{\sqrt{3}}{2}\cdot\sin x$

24) $\cos\left(\frac{\pi}{6}-x\right)$ $\frac{\sqrt{3}}{2}\cdot\cos x+\frac{1}{2}\cdot\sin x$

25) $\sin\left(\frac{\pi}{3}+x\right)+\cos\left(\frac{\pi}{6}+x\right)$
$\sqrt{3}\cos x$

26) $\sin\left(\frac{\pi}{3}-x\right)-\cos\left(\frac{\pi}{6}-x\right)$
$-\sin x$

27) $\cos(60+x)-\sin(120+x)$ $\frac{1-\sqrt{3}}{2}(\cos x+\sin x)$

28) $\sin\left(x+\frac{2}{3}\pi\right)-\cos\left(\frac{\pi}{6}+x\right)$
0

29) $\cos\left(x-\frac{11}{6}\pi\right)-\sin\left(\frac{5}{3}\pi+x\right)$
$\sqrt{3}\cos x-\sin x$

30) $\frac{\sin(x+45)-\cos(x+45)}{\sin(x+135)+\cos(x+315)}$
$\tan x$

Using the cofunction identities, find the measure of the acute angle x

31) $\sin x=\cos 50$
$x=40°$

32) $\cos x=\sin\frac{\pi}{3}$ $x=\frac{\pi}{6}$

33) $\tan x=\cot 17$
$x=73°$

34) $\sec x=\csc\frac{\pi}{5}$ $x=\frac{3}{10}\pi$

Simplify the following

35) $-\sin x\sin(-x)+\cos x\cos(-x)$
1

36) $\tan(-x)-2\cos(-x)+2\sin(-x)\cot(-x)$
$-\tan x$

37) $\sin(-x)+\cos(-x)+\sec(-x)+$
$+2\sin x-\sec x+3\cos x$

$\sin x+4\cos x$

38) $\sin^2 -x\ +\cos^2 -x+\tan x\cot(-x)$

0

Simplify the following

39) $\cos 2x+\sin 2x\tan x$

1

40) $\tan 2x\cdot(1+\tan x)\cot x$ $\dfrac{2}{1-\tan x}$

41) $\cos 2x+\sin 2x+2\sin^2 x$

$(\cos x+\sin x)^2$

42) $\cos 2x-\sin 2x+(\sin x+\cos x)^2$

$2\cos^2 x$

43) $\cos 2x-\dfrac{\cos x\sin 2x}{\sin x}$

-1

44) $\dfrac{\sin 2x}{1+\cos 2x}-\tan x$

0

45) $\dfrac{1-\cos 2x}{1+\cos 2x}\cdot\cot x$

$\tan x$

46) $\sin 4x$

$4(\sin x\cos^3 x-\sin^3 x\cos x)$

3.4 Trigonometric Equations

Topics: solving any trigonometric equations

Review

Simplify $\dfrac{\tan x+\sin x}{\tan x-\sin x}$ $\left[(\csc x+\cot x)^2\right]$

Evaluate $2\sin 210^\circ-3\cos 135^\circ+4\tan 330^\circ-2\csc 225^\circ$ $\left[\dfrac{7\sqrt{2}}{2}-\dfrac{4\sqrt{3}}{3}-1\right]$

Simplify $\dfrac{1+\cos 2x}{\sin 2x}$ $[\cot x]$

Simplify $\dfrac{1-\cos 2x}{\sin 2x}\cdot\tan\left(\dfrac{\pi}{2}-x\right)+\sin^2(-x)$ $\left[1+\sin^2 x\right]$

Fundamental Strategy

Most trigonometric equations can be solved using the following four-step process.

Step 1

Isolate the trigonometric function.

Step 2

Determine the reference angle.

Step 3

Identify the quadrants where the function has the required sign.

Sine is positive in Quadrants I and II.

Cosine is positive in Quadrants I and IV.

Tangent is positive in Quadrants I and III.

Step 4

Write all solutions in the required interval.

Always check that every answer lies inside the stated interval.

Example 1

Solve $-4+3\tan x=-4;\quad 0\le x<360^\circ$

Solution:

$-4+3\tan x=-4$

$-3\tan x=0$

$\tan x=0$

$x=0^\circ,180^\circ$

Example 2

Solve $-4+2\cos x=0;\quad 0\le x<360^\circ$

Solution:

$-4+2\cos x=0$

$2\cos x=4$

$\cos x=2$

Since $-1\le\cos x\le 1$ there is no real angle satisfying this equation. No Solution.

Example 3

Solve $\csc^2 x = 4; \quad 0 \le x < 360^\circ$

Solution:

$\csc x = \dfrac{1}{\sin x}$ then $\dfrac{1}{\sin^2 x} = 4$ then

$\sin^2 x = \dfrac{1}{4}$

$\sin x = \pm\sqrt{\dfrac{1}{4}} = \pm\dfrac{1}{2}$

The reference angle is 30°. Sine is positive in Quadrant I and II and negative in Quadrant III and IV. So we have 4 solutions:

$x = 30^\circ$

$x = 180^\circ - 30^\circ = 150^\circ$

$x = 180^\circ + 30^\circ = 210^\circ$

$x = 360^\circ - 30^\circ = 345^\circ$

Example 4

Solve $3 + 2\tan x = 1; \quad 0 \le x < 360^\circ$

Solution:

$3 + 2\tan x = 1$

$2\tan x = -2$

$\tan x = -1$

The reference angle is 45°. Since the tangent is negative in Quadrant II and IV

$x = 180^\circ - 45^\circ = 135^\circ$

$x = 360^\circ - 45^\circ = 315^\circ$

Example 5

Solve $1 + 4\cos(-x) = -1; \quad 0 \le x < 2\pi$

Solution:

Because cosine is an even function, $\cos(-x) = \cos x$ then

$1 + 4\cos x = -1$

$4\cos x = -2$

$\cos x = -\dfrac{1}{2}$

The reference angle is $\dfrac{\pi}{3}$. Cosine is negative in Quadrant II and III, therefore

$x = \pi - \dfrac{\pi}{3} = \dfrac{2}{3}\pi$

$x = \pi + \dfrac{\pi}{3} = \dfrac{4}{3}\pi$

Example 6

Solve $\log_2(\sin x)=0;\quad 0\le x<2\pi$

Solution:

$\log_2(\sin x)=0$, using the definition $\log_b a=x\leftrightarrow b^x=a$ we have $\sin x=2^0=1$

$\sin x=1$ only when $x=\dfrac{\pi}{2}$

Example 7

Solve $-4+\sec x=-6;\quad 0\le x<2\pi$

Solution:

$-4+\sec x=-6$

$\sec x=-6+4$

$\sec x=-2$

$\dfrac{1}{\cos x}=-2$

$\cos x=-\dfrac{1}{2}$

The reference angle is $\dfrac{\pi}{3}$. Cosine is negative in Quadrant II and III, therefore

$x=\pi-\dfrac{\pi}{3}=\dfrac{2}{3}\pi$

$x=\pi+\dfrac{\pi}{3}=\dfrac{4}{3}\pi$

Solving Equations with Angle Shifts and Multiple Angles

In the previous section, we focused on equations involving a single angle variable, x, within standard trigonometric functions. We now expand our toolkit to tackle equations where the argument of the trigonometric function is a shifted or scaled version of the variable, such as $\sin\left(x+\dfrac{\pi}{3}\right)$, $\cos(2x)$ or $\tan\left(\dfrac{x}{2}\right)$

Solving these equations introduces a critical two-step process: first, we must isolate the trigonometric function of the composite angle and solve for that entire angle expression over the fundamental period.

Example 8

Solve $\dfrac{-4+\sqrt{3}}{4}=-1+\dfrac{1}{2}\cos\left(x+210^\circ\right);\quad 0\le x<360^\circ$

Solution:

Step 1: Isolate cosine

$\dfrac{-4+\sqrt{3}}{4}=-1+\dfrac{1}{2}\cos\left(x+210^\circ\right)$

$-1+\dfrac{\sqrt{3}}{4}=-1+\dfrac{1}{2}\cos\left(x+210^\circ\right)$

$$\frac{\sqrt{3}}{4}=\frac{1}{2}\cos\left(x+210^\circ\right)$$

$$\frac{\sqrt{3}}{2}=\cos\left(x+210^\circ\right)$$

Step 2: solve for $x+210^\circ$

We need angles $\alpha=x+210^\circ$ such that $\cos\alpha=\frac{\sqrt{3}}{2}$. Cosine is positive in quadrant I and IV; it has period 360° and equals $\frac{\sqrt{3}}{2}$ at: $\alpha=30^\circ+360^\circ k$ *or* $\alpha=330^\circ+360^\circ k$ where *k* is an integer. Then solving for *x*:

$x+210^\circ=30^\circ+360^\circ k \rightarrow x=-180^\circ+360^\circ k$

$x+210^\circ=330^\circ+360^\circ k \rightarrow x=120^\circ+360^\circ k$

For *k = 0*

$x=-180^\circ+360^\circ k=-180^\circ\ \left(\text{not in domain}\right)$

$x=120^\circ+360^\circ k=120^\circ$

For *k = 1*

$x=-180^\circ+360^\circ k=180^\circ$

$x=120^\circ+360^\circ k=480^\circ\ \left(\text{not in domain}\right)$

Therefore solution is: $x=120^\circ,180^\circ$

Example 9

Solve $4+\frac{2}{3}\sin\frac{x}{2}=\frac{12+\sqrt{3}}{3};\quad 0\le x<360^\circ$

Solution:

Step 1: Isolate sine

$$4+\frac{2}{3}\sin\frac{x}{2}=\frac{12+\sqrt{3}}{3}$$

$$4+\frac{2}{3}\sin\frac{x}{2}=4+\frac{\sqrt{3}}{3}$$

$$\frac{2}{3}\sin\frac{x}{2}=\frac{\sqrt{3}}{3}$$

$$\sin\frac{x}{2}=\frac{\sqrt{3}}{2}$$

Step 2: solve for $\frac{x}{2}$

We need angles $\alpha=\frac{x}{2}$ such that $\sin\alpha=\frac{\sqrt{3}}{2}$. Sine is positive in quadrant I and II; it has period 360° and equals $\frac{\sqrt{3}}{2}$ at: $\alpha=60^\circ+360^\circ k$ *or* $\alpha=120^\circ+360^\circ k$ where *k* is an integer. Then solving for $\frac{x}{2}$:

$\frac{x}{2}=60^\circ+360^\circ k \rightarrow x=120^\circ+720^\circ k$; $\frac{x}{2}=120^\circ+360^\circ k \rightarrow x=240^\circ+720^\circ k$

For *k* = *0*

$x = 120^\circ + 720^\circ k = 120^\circ$

$x = 240^\circ + 720^\circ k = 240^\circ$

For *k* = *1*

not in domain

Therefore solution is: $x = 120^\circ, 240^\circ$

Example 10

Solve $5 - \frac{2}{\sqrt{3}} = 5 - \frac{2}{3}\tan\left(2x - \frac{\pi}{6}\right); \quad 0 \le x < 2\pi$

Solution:

Step 1: Isolate tangent

$$5 - \frac{2}{\sqrt{3}} = 5 - \frac{2}{3}\tan\left(2x - \frac{\pi}{6}\right)$$

$$-\frac{2}{\sqrt{3}} = -\frac{2}{3}\tan\left(2x - \frac{\pi}{6}\right)$$

$$\frac{1}{\sqrt{3}} = \frac{1}{3}\tan\left(2x - \frac{\pi}{6}\right)$$

$$\frac{3}{\sqrt{3}} = \tan\left(2x - \frac{\pi}{6}\right)$$

$$\tan\left(2x - \frac{\pi}{6}\right) = \sqrt{3}$$

Step 2: solve for $2x - \frac{\pi}{6}$

We need angles $\alpha = 2x - \frac{\pi}{6}$ such that $\tan\alpha = \sqrt{3}$. Tangent is positive in quadrant I and III; it has period π and equals $\sqrt{3}$ at: $\alpha = \frac{\pi}{3} + \pi k \ \ or \ \alpha = \frac{4}{3}\pi + \pi k$ where *k* is an integer. Then solving for $2x - \frac{\pi}{6}$:

$$2x - \frac{\pi}{6} = \frac{\pi}{3} + \pi k \quad \rightarrow \ x = \frac{\pi}{4} + \frac{\pi}{2}k$$

$$2x - \frac{\pi}{6} = \frac{4}{3}\pi + \pi k \ \rightarrow \ x = \frac{3}{4}\pi + \frac{\pi}{2}k$$

For *k* = *0*

$x = \frac{\pi}{4} + \frac{\pi}{2}k = \frac{\pi}{4}$

$x = \frac{3}{4}\pi + \frac{\pi}{2}k = \frac{3}{4}\pi$

$x = \frac{\pi}{4} + \frac{\pi}{2}k = \frac{3}{4}\pi$

$x = \frac{3}{4}\pi + \frac{\pi}{2}k = \frac{5}{4}\pi$

For *k* = *1*

$x = \frac{\pi}{4} + \frac{\pi}{2}k = \frac{3}{4}\pi$

$x = \frac{3}{4}\pi + \frac{\pi}{2}k = \frac{5}{4}\pi$

For $k = 2$

$$x = \frac{\pi}{4} + \frac{\pi}{2}k = \frac{5}{4}\pi$$

$$x = \frac{3}{4}\pi + \frac{\pi}{2}k = \frac{7}{4}\pi$$

For $k = 3$

$$x = \frac{\pi}{4} + \frac{\pi}{2}k = \frac{7}{4}\pi$$

$$x = \frac{3}{4}\pi + \frac{\pi}{2}k = \frac{9}{4}\pi \text{ (outside of the domain)}$$

Therefore solution is: $x = \frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$

Example 11

Solve $5 - 6\sin(2x) = 8; \quad 0 \le x < 2\pi$

Solution:

Step 1: Isolate sine

$$5 - 6\sin(2x) = 8$$

$$-6\sin(2x) = 3$$

$$\sin 2x = -\frac{1}{2}$$

Step 2: solve for $2x$

We need angles $\alpha = 2x$ such that $\sin\alpha = -\frac{1}{2}$. Sine is negative in quadrant III and IV; it has period 2π and equals $-\frac{1}{2}$ at: $\alpha = \frac{7}{6}\pi + 2\pi k \text{ or } \alpha = \frac{11}{6}\pi + 2\pi k$ where k is an integer. Then solving for $2x$:

$$2x = \frac{7}{6}\pi + 2\pi k \rightarrow x = \frac{7}{12}\pi + \pi k$$

$$2x = \frac{11}{6}\pi + 2\pi k \rightarrow x = \frac{11}{12}\pi + \pi k$$

For $k = 0$

$$x = \frac{7}{12}\pi + \pi k = \frac{7}{12}\pi$$

$$x = \frac{11}{12}\pi + \pi k = \frac{11}{12}\pi$$

For $k = 1$

$$x = \frac{7}{12}\pi + \pi k = \frac{19}{12}\pi$$

$$x = \frac{11}{12}\pi + \pi k = \frac{23}{12}\pi$$

For $k = 2$

(outside of the domain)

Therefore solution is: $x = \frac{7}{12}\pi, \frac{11}{12}\pi, \frac{19}{12}\pi, \frac{23}{12}\pi$

Advanced Trigonometric Equations

As we have seen from the solution strategies developed earlier, trigonometric equations are solved in the same way as algebraic equations. The following exercises begin with straightforward quadratic forms in a single function and advance to equations requiring the strategic use of Pythagorean identities, factoring by grouping, and the conversion of logarithmic forms. As you work through these problems, pay close attention to the necessary domain restrictions and the importance of checking for extraneous solutions, particularly when squaring both sides or dealing with functions like secant and tangent.

Example 12

Solve $2\sin^2 x - \sin x = 0; \quad 0 \le x < 2\pi$

Solution:

Factor: $\sin x(2\sin x - 1) = 0$

So, $\sin x = 0$ or $\sin x = \frac{1}{2}$

For $\sin x = 0 \rightarrow x = 0, \pi$; for $\sin x = \frac{1}{2} \rightarrow x = \frac{\pi}{6}$

Example 13

Solve $2\cos^2 x - 3\cos x + 1 = 0; \quad 0 \le x < 2\pi$

Solution:

Factor: $(2\cos x - 1)(\cos x - 1) = 0$

So, $\cos x = 1$ or $\cos x = \frac{1}{2}$

For $\cos x = 1 \rightarrow x = 0$; for $\cos x = \frac{1}{2} \rightarrow x = \frac{\pi}{3}$

Example 14

Solve $\sin x \tan x = \sin x; \quad 0 \le x < 2\pi$

Solution:

$\sin x \tan x - \sin x = 0$

Factor: $\sin x(\tan x - 1) = 0$

So, $\sin x = 0$ or $\tan x = 1$

For $\sin x = 1 \rightarrow x = \frac{\pi}{2}$; for $\tan x = 1 \rightarrow x = \frac{\pi}{4}$ and $\frac{5}{4}\pi$

NOTE: Losing a Root

In the example above the factor $\sin x$ on both sides of the equation was **NOT** canceled. Doing so would have caused the loss of a solution. A simple way to avoid losing roots is to bring all terms to one side of the equation and factor.

Example 15

Solve $\tan^2 x - \sec x - 1 = 0; \quad 0 \le x < 2\pi$

Solution:

Use identity $\tan^2 x = \sec^2 x - 1$

Substitute: $(\sec^2 - 1) - \sec x - 1 = 0 \rightarrow \sec^2 x - \sec x - 2 = 0$

Factor: $(\sec x + 1)(\sec x - 2) = 0 \rightarrow$

So, $\sec x = -1 \rightarrow \cos x = -1$ or $\sec x = 2 \rightarrow \cos x = \frac{1}{2}$

For $\cos x = -1 \rightarrow x = \pi$; for $\cos x = \frac{1}{2} \rightarrow x = \frac{\pi}{3}$ and $\frac{5}{3}\pi$

Example 16

Solve $\cos 2x - 3\sin - 2 = 0; \quad 0 \le x < 2\pi$

Solution:

Use identity $\cos 2x = 1 - 2\sin^2 x$

Substitute: $\left(1 - 2\sin^2 x\right) - 3\sin - 2 = 0 \rightarrow 2\sin^2 x + 3\sin x + 1 = 0$

Factor: $\left(2\sin x + 1\right)\left(\sin x + 1\right) = 0$; So, $\sin x = -\frac{1}{2}$ or $\sin x = -1$

For $\sin x = -1 \rightarrow x = \frac{3}{2}\pi$; for $\sin x = -\frac{1}{2} \rightarrow x = \frac{7}{6}\pi$ and $\frac{11}{6}\pi$

Example 17

Solve $\csc^2 x - 2\cot x = 0; \quad 0 \le x < 2\pi$

Solution:

Use identity $\csc^2 x = 1 + \cot^2 x$

Substitute: $\left(1 + \cot^2 x\right) - 2\cot x = 0 \rightarrow \cot^2 x - 2\cot x + 1 = 0$

Factor: $\left(\cot x - 1\right)^2 = 0$

So, $\cot x = 1 \rightarrow \tan x = 1$; for $\tan x = 1 \rightarrow x = \frac{\pi}{4}, \frac{5}{4}\pi$

Example 18

Solve $\sin^4 x - \cos^4 x = \frac{1}{2}; \quad 0 \le x < 2\pi$

Solution:

Factor: $\left(\sin^2 x - \cos^2 x\right)\left(\sin^2 x + \cos^2 x\right) = \frac{1}{2}$

Since $\sin^2 x + \cos^2 x = 1$ we have $\sin^2 x - \cos^2 x = \frac{1}{2}$

Use identity $\cos^2 x = 1 - \sin^2 x$

Substitute: $\sin^2 x - \left(1 - \sin^2 x\right) = \frac{1}{2} \rightarrow 2\sin^2 x = \frac{3}{2} \rightarrow \sin^2 x = \frac{3}{4}$

So, $\sin x = \pm\frac{\sqrt{3}}{2} \rightarrow x = \frac{\pi}{3}, \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{5}{3}\pi$

Example 19

Solve $\ln\left(1 - 2\cos^2 x\right) = 0; \quad 0 \le x < 2\pi$

Solution:

Since $\ln\left(y\right) = 0$ when $y = 1 \rightarrow \ 1 - 2\cos^2 x = 1 \ \rightarrow 2\cos^2 x = 0 \rightarrow \cos x = 0$

For $\cos x = 0 \rightarrow x = \frac{\pi}{2}, \frac{3}{2}\pi$

Practice Problems

Solve each equation for $0 \le \theta < 360$.

1) $-4 - \tan\theta = -4$

$\{0, 180\}$

2) $-3 + 3\tan\theta = 0$

$\{45, 225\}$

3) $-5 + 2\cos\theta = -6$

$\{120, 240\}$

4) $-4 - \frac{1}{4} \cdot \cos\theta = -4$

$\{90, 270\}$

5) $5 + 4\sin\theta = 3$

$\{210, 330\}$

6) $-4 - 4\cos\frac{\theta}{2} = -2$

$\{240\}$

7) $0 = -3 - 3\tan(\theta + 90)$

$\{45, 225\}$

8) $-5 - 2\tan 2\theta = -3$

$\left\{67\frac{1}{2}, 157\frac{1}{2}, 247\frac{1}{2}, 337\frac{1}{2}\right\}$

9) $4 - 2\sin(\theta + 150) = 5$

$\{60, 180\}$

Solve each equation for $0 \le \theta < 2\pi$.

10) $\cos\theta = \frac{\sqrt{3}}{2}$

$\left\{\frac{\pi}{6}, \frac{11\pi}{6}\right\}$

11) $\sin\theta = \frac{\sqrt{3}}{2}$

$\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$

12) $\sin\theta = -\frac{1}{2}$

$\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

13) $1 = \cos\theta$

$\{0\}$

Solve each equation for $0 \le \theta < 360$.

14) $-\frac{13}{5} - \sin\frac{\theta}{2} = -3 - \frac{4}{5} \cdot \sin\frac{\theta}{2}$

No solution.

15) $2 - 3\tan -4\theta = -3\sqrt{3} + 2$

$\{30, 75, 120, 165, 210, 255, 300, 345\}$

16) $-3\sqrt{3} - 2 + 2\tan(\theta + 270) = -2 - \tan(\theta + 270)$

$\{150, 330\}$

17) $5 + 3\sin(\theta + 225) = 5 + \sin(\theta + 225)$

$\{135, 315\}$

Solve each equation for $0 \le \theta < 2\pi$.

18) $\frac{3}{2} - \sec\left(\theta + \frac{\pi}{4}\right) = 1 - \frac{3}{2} \cdot \sec\left(\theta + \frac{\pi}{4}\right)$ $\left\{\frac{3\pi}{4}\right\}$

19) $-2\csc\left(\theta + \frac{\pi}{6}\right) = 2 - \csc\left(\theta + \frac{\pi}{6}\right)$ $\left\{\pi, \frac{5\pi}{3}\right\}$

20) $-8 - 2\cot -2\theta = -5 + \cot -2\theta$

$\left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\right\}$

21) $\sqrt{3} - 7 = -7 + \cot\left(\theta + \frac{5\pi}{4}\right)$

$\left\{\frac{11\pi}{12}, \frac{23\pi}{12}\right\}$

Solve each equation for $0 \le \theta < 360$.

22) $3 - 4\sec -4\theta = -3 - \sec -4\theta$

$\{15, 75, 105, 165, 195, 255, 285, 345\}$

23) $\frac{4}{3} + 2\tan -\theta = 1 + \frac{5}{3} \cdot \tan -\theta$

$\{45, 225\}$

24) $5 - 2\sec(\theta + 90) = 5 + \sec(\theta + 90)$

No solution.

25) $2\sqrt{3} - 4 = -4 + 2\tan\frac{\theta}{3}$

$\{180\}$

Solve each equation for $0 \le \theta < 2\pi$.

26) $-4 + \frac{5}{2} \cdot \tan 3\theta = \frac{-8 + \sqrt{3}}{2} + 2\tan 3\theta$

$\left\{\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}\right\}$

27) $5 - 3\cos -\theta = -2\sqrt{2} + 5 - \cos -\theta$

No solution.

28) $2 + 4\sin\frac{\theta}{2} = -2\sqrt{3} + 2$

No solution.

29) $-3 - 5\sin\frac{\theta}{3} = -2\sin\frac{\theta}{3}$

No solution.

30) $3 - 2\cos^2 x - 3\sin x = 0$ $\frac{\pi}{6}, \frac{5}{6}\pi, \frac{\pi}{2}$

31) $\sin x + \cos x = 1$ $x = 0, \frac{\pi}{2}$

32) $2 - 3\sin^2 x - \cos x = 0$

$0, 2\pi$

33) $\sin\left(x + \frac{\pi}{6}\right) = \cos\left(x - \frac{\pi}{3}\right)$ $x = \frac{\pi}{3}, \frac{4}{3}\pi$

34) $5 - 4\cos^2 x - 4\sin x = 0$ $\frac{\pi}{6}, \frac{5}{6}\pi$

35) $\sin\frac{x}{2} = \cos\frac{x}{2}$ $\frac{\pi}{4}, \frac{5}{4}\pi$

36) $3\tan^3 x - 3\tan^2 x - \tan x + 1 = 0$ $\frac{\pi}{6}, \frac{\pi}{4}, \frac{5}{6}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi$

37) $2\sin x - 2\sqrt{3}\cos x - \sqrt{3}\tan x + 3 = 0$ $\frac{\pi}{6}, \frac{\pi}{3}, \frac{4}{3}\pi, \frac{11}{6}\pi$

38) $\ln(2 - \sin^2 x) = 0$ $\frac{\pi}{2}, \frac{3}{2}\pi$

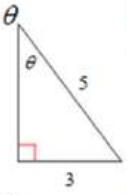
39) $\log_3(2\sin x) = \frac{1}{2}$ $\frac{\pi}{3}, \frac{2}{3}\pi$

40) $\cos 2x = \sin x$ $\frac{\pi}{6}, \frac{5}{6}\pi, \frac{3}{2}\pi$

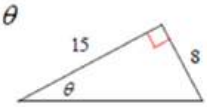
CHAPTER 3-Review

Find the value of the trig function indicated.

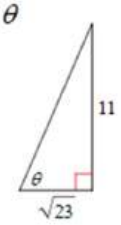
1) $\cos\theta$ $\frac{4}{5}$



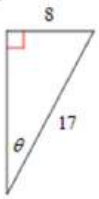
2) $\cos\theta$ $\frac{15}{17}$



3) $\csc\theta$ $\frac{12}{11}$



4) $\sec\theta$ $\frac{17}{15}$



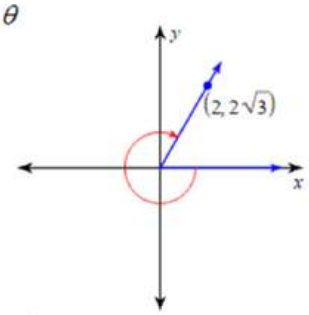
5) Find $\cot\theta$ if $\tan\theta = \frac{12}{5}$ $\frac{5}{12}$

6) Find $\csc\theta$ if $\sec\theta = \frac{25}{24}$ $\frac{25}{7}$

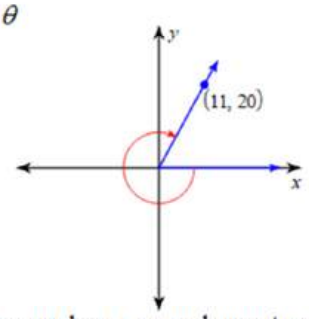
7) Find $\cot\theta$ if $\cos\theta = \frac{2\sqrt{10}}{7}$ $\frac{2\sqrt{10}}{3}$

Use the given point on the terminal side of angle θ to find the value of the trigonometric function indicated.

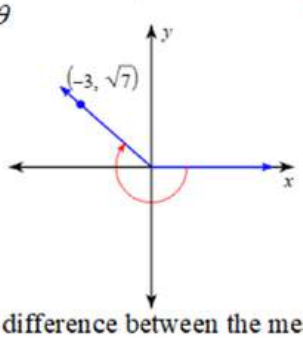
8) $\sin\theta$ $\frac{\sqrt{3}}{2}$



9) $\tan\theta$ $\frac{20}{11}$



10) $\sin\theta$ $\frac{\sqrt{7}}{4}$



11) Two angles are supplementary. One angle is 51 degrees less than twicw the other. Find the measure of both angles.

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12) The difference between the measures of two complementary angles is $\frac{1}{10}\pi$. Find the measure of each angle.

$\frac{3}{10}\pi$ and $\frac{1}{5}\pi$

13) Angle A and angle B are supplementary. Angle A is $\frac{\pi}{3}$ and angle B is $\frac{2}{3}\pi$. If angle C is complementary to angle A, find the measure of angle C. $C=\frac{\pi}{6}$

Find the amplitude, the period in radians, the phase shift, the minimum and maximum values.

14) $y=-2+9\sin\left(7\theta+\frac{5\pi}{3}\right)$

Amplitude: 9

Period: $\frac{2\pi}{7}$

Phase shift: Left $\frac{5\pi}{21}$

Min: −11

Max: 7

15) $y=-1+2\sin\left(2\theta+\frac{5\pi}{6}\right)$

Amplitude: 2

Period: π

Phase shift: Left $\frac{5\pi}{12}$

Min: −3

Max: 1

16) $y=\frac{1}{4}\cdot\cos\left(7\theta-\frac{\pi}{2}\right)+2$ Amplitude: $\frac{1}{4}$

Period: $\frac{2\pi}{7}$

Phase shift: Right $\frac{\pi}{14}$

Min: $\frac{7}{4}$

Max: $\frac{9}{4}$

17) $y=6\cos\left(\frac{\theta}{3}+\frac{\pi}{4}\right)-5$ Amplitude: 6

Period: 6π

Phase shift: Left $\frac{3\pi}{4}$

Min: −11

Max: 1

18) A company's monthly revenue $R(t)$ (in thousands of dollars) is modeled by $R(t)=8\cos(45t+90)+20$ where $1\le t\le 12$ represents months ($t=1=$ January).

(a) Rewrite in the form $R(t)=A\cos(B(t-h))+C$ and identify all parameters.

$R(t)=8\cos(45(t+2))+20;\ A=8, B=45, h=-2, C=20$

(b) How many full cycles occur in the 12-month period? Period = 8months;cycles in 12 months $=\frac{12}{8}=1.5$

(c) What is the revenue in October? $R(10)=\$12000$

(d) In what month(s) does the revenue equal exactly \$24,000? no integer months, so no month has exactly \$24000

19) A company's quarterly profit $Q(t)$ (in hundreds of dollars) is modeled by $Q(t)=15\sin(30t-120)+25$ where $1\le t\le 8$ represents quarters ($t=1=$ Q1).

(a) Rewrite in the form $Q(t)=A\sin(B(t-h))+C$ and identify all parameters.

$Q(t)=15\sin(30(t-4))+25;\ A=15, B=30, h=4, C=25$

(b) How many full cycles occur in the 8-quarter period?

Period = 12quarters;cycles in 8 months $=\frac{8}{12}=0.66...$so 0 full cycles

(c) What is the profit in Q5? $Q(5)=\$3250$

(d) In what quarter(s) does the profit equal exactly \$40,000? $Q(7)$ has profit exactly \$40000

20) Evaluate $3\sin 30^\circ+2\cos 60^\circ-(\sec 45^\circ)(\sin 45^\circ)+\tan 45^\circ-\cos 0^\circ$ $\frac{3}{2}$

21) Evaluate $4\cos\frac{\pi}{3}-\csc\frac{\pi}{6}+\sqrt{3}\tan\frac{\pi}{3}-2\sin\frac{\pi}{2}+\sec 0$ 2

22) Evaluate $\tan 45^\circ+\left(3\cos 30^\circ\right)\left(\sin 60^\circ\right)-2\sin^2 45^\circ+\sec 60^\circ-5$ $-\frac{3}{4}$

23) Evaluate $\sin 90^\circ+2\cos 180^\circ-3\tan 0^\circ+4\sin^2 270^\circ-\cos 270^\circ+\cot 45^\circ$ 4

24) Simplify $\frac{\sin^2 x}{1-\sin^2 x}-\cot x+\frac{1-\cos^2 x}{\sin^2 x}-\frac{1}{\cos^2 x}$ $-\cot x$

25) Simplify $\cos x\tan x+\sin x\left(1-\cos x\right)+\cot x\sin^2 x$ $2\sin x$

26) Simplify $\left(\frac{1}{\sin^2 x}+\cot^2 x\right)\cdot\frac{1}{\csc^2 x}$ $1+\cos^2 x$ or $2-\sin^2 x$

27) Simplify $\frac{\sin 2x}{1+\cos 2x}+\tan x-\frac{\sin^2 x}{\cos x\tan x}$ $2\tan x-\sin x$

28) Simplify $\frac{\cos 2x}{\sin x+\cos x}+\sin x-\cos x$ 0

29) Simplify $\sin\left(x+30^\circ\right)+\sin\left(x-30^\circ\right)-\sqrt{3}\cos x$ $\sqrt{3}\left(\sin x-\cos x\right)$

30) Simplify $\frac{\sin 2x}{1-\cos 2x}\cdot\tan x+\cos 2x+2\sin^2 x$ 2

31) Simplify $\sin\left(x+60^\circ\right)-\sin\left(x-60^\circ\right)-\sqrt{3}\cos x$ 0

32) Simplify $\cos\left(x+45^\circ\right)+\cos\left(x-45^\circ\right)-\sqrt{2}\cos x$ 0

33) Solve $\sin 2x=\cos x;\ 0^\circ\le x<360^\circ$ $30^\circ,90^\circ,150^\circ,270^\circ$

34) Solve $\cos 2x=\sin x;\ 0^\circ\le x<360^\circ$ $30^\circ,150^\circ,270^\circ$

35) Solve $2\cos x-\cos x-1=0;\ 0^\circ\le x<360^\circ$ $0^\circ,120^\circ,240^\circ$

36) Solve $\cos 2x+5\cos x+3=0;\ 0^\circ\le x<360^\circ$ $120^\circ,240^\circ$

37) Solve $\sin 2x=\sqrt{2}\cos x;\ 0\le x<2\pi$ $\frac{\pi}{4},\frac{\pi}{2},\frac{3}{4}\pi,\frac{3}{2}\pi$

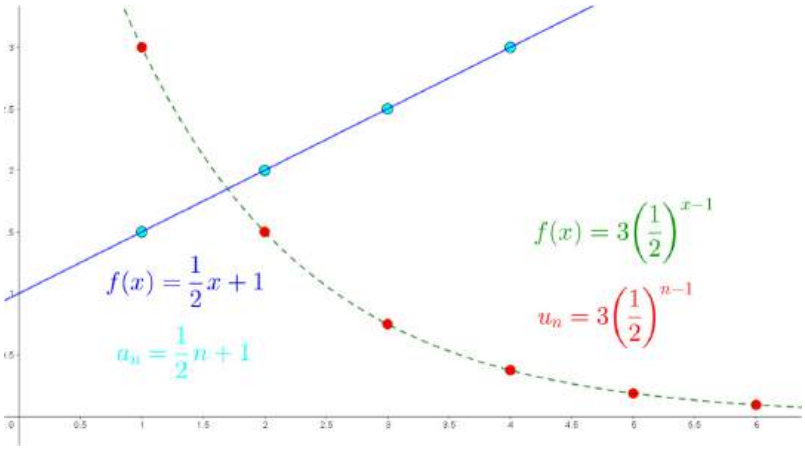
38) Solve $\cos 2x=\cos x;\ 0\le x<2\pi$ $0,\frac{2}{3}\pi,\frac{4}{3}\pi$

39) Solve $\sin 2x+\sin x=0;\ 0\le x<2\pi$ $0,\frac{2}{3}\pi,\pi,\frac{4}{3}\pi$

40) Solve $\cos 2x+3\sin x=2;\ 0\le x<2\pi$ $\frac{\pi}{6},\frac{\pi}{2},\frac{5}{6}\pi$

CHAPTER 4 – Sequences and Series

"Infinite series are the bridge between the finite world of arithmetic and the infinite landscape of calculus."— Morris Kline



Review

1. Given $f(x)=\dfrac{3x+2}{x-5},\ x\neq 5$

(i) find $f^{-1}(x)$ — $f^{-1}(x)=\dfrac{5x+2}{x-3},\ x\neq 3$

(ii) Determine the domain and range of $f^{-1}(x)$ — all real numbers except 3, $x\neq 3$

(iii) Solve $f(x)=f^{-1}(x)$ — $x=4\pm 3\sqrt{2}$

2. Solve $4^{x+1}-3\left(2^{x+2}\right)+8=0$ — $x=0,1$

3. Solve $\log_2(x+3)+\log_2(x-1)=\log_2(4x+12)$ — $x=5$

4.1 Arithmetic Sequences

Topic: Definition, Recursive and Explicit Formula

In the world around us, change rarely happens in a single, isolated event. Instead, we observe patterns, the steady growth of a savings account, the gradual depreciation of a car, or the consistent increase in monthly rent. To describe and predict these step-by-step changes, mathematicians developed the concepts of *sequences* and *series*. A *sequence* is an ordered list of numbers that reveals a pattern, acting as a roadmap for how a quantity evolves over discrete intervals. A *series* takes this a step further; it is the sum of the terms of a sequence, allowing us to measure the total cumulative effect of that pattern. From calculating compound interest in business to modeling population growth in biology and building efficient algorithms in computer science, these mathematical tools are the engines that help us understand cumulative progress and forecast future outcomes.

DEFINITION SEQUENCE

A sequence s_n is a **function** whose domain consists of consecutive positive integers $(n = 1, 2, 3, ...)$.

Each corresponding value $s_1, s_2, s_3, ...$ is called a **term** of the sequence.

$s_1 = \text{first term}$

$s_2 = \text{second term and so on}$

The number written subscript to the right of each term is called **index**.

The general term, the $n(th)$ term, of a sequence is denoted by the symbol s_n.

A sequence that has a limited number of terms is called **finite** sequence.
A sequence that has an unlimited number of terms is called **infinite** sequence.

EXAMPLE 1

Find the next three terms of the sequence: $26, 14, 8, 5, \frac{7}{2}, ...$

Solution:

Let's examine the pattern by finding the differences between consecutive terms:

$26 - 14 = 12$

$14 - 8 = 6$

$8 - 5 = 3$

$5 - \frac{7}{2} = \frac{3}{2}$

Each difference is half of the previous difference, so the next differences will be:

$\frac{3}{2} \div 2 = \frac{3}{4}$

$\frac{3}{4} \div 2 = \frac{3}{8}$

$\frac{3}{8} \div 2 = \frac{3}{16}$

So the next three terms:

$\frac{7}{2} - \frac{3}{4} = \frac{11}{4}$

$\frac{11}{4} - \frac{3}{8} = \frac{19}{8}$

$\frac{19}{8} - \frac{3}{16} = \frac{35}{16}$

EXAMPLE 2

Find the next three terms of the sequence: $\frac{7}{4},\frac{25}{12},\frac{29}{12},\frac{11}{4},\frac{37}{12},...$

Solution:

The denominators are 4 and 12. The least common denominator is 12. So the sequence in twelfth is

$\frac{21}{12},\frac{25}{12},\frac{29}{12},\frac{33}{12},\frac{37}{12},...$

The numerators are : 21, 25,29,33, 37,…

They have a common difference of 4. So the next numerators are:

37 + 4 = 41, 41+4= 45, 45 + 4 = 49

The next three terms are: $\frac{41}{12},\frac{45}{12},\frac{49}{12}$. Simplify: $\frac{41}{12},\frac{15}{4},\frac{49}{12}$

EXAMPLE 3

Find the next three terms of the sequence: $\frac{1}{2},1,\frac{9}{8},1,\frac{25}{32},...$

Solution:

Let's express each term as a fraction: $\frac{1}{2},\frac{4}{4},\frac{9}{8},\frac{16}{16},\frac{25}{32},...$

Numerators are perfect squares: $1^2,2^2,3^2,4^2,5^2,...$

Denominators are powers of 2: $2^1,2^2,2^3,2^4,2^5,...$

The next three terms are: $\frac{6^2}{2^6}=\frac{36}{64};\frac{7^2}{2^7}=\frac{49}{128};\frac{8^2}{2^8}=\frac{64}{256}$. Simplify: $\frac{9}{16},\frac{49}{128},\frac{1}{4}$

EVALUATING A SEQUENCE USING AN EXPLICIT FORMULA

An explicit formula defines the n-th term of a sequence as a function of n, allowing you to find any term directly without needing to know the previous terms.

To evaluate a sequence using an explicit formula, follow these steps:

Identify the explicit formula. It is typically written in the form $a_n = f(n)$, where n represents the position of the term (1st, 2nd, 3rd, etc.).

Substitute the desired index. Replace n with the specific positive integer corresponding to the term you wish to find. For example, to find the 10th term, substitute $n = 10$.

Simplify the expression. Perform the arithmetic operations to obtain the value of the term.

EXAMPLE 4

Given the general term: $a_n = 2^{n-1}$, find the first four terms of the sequence

Solution:

$a_1 = 2^{1-1} = 2^0 = 1$

$a_2 = 2^{2-1} = 2^1 = 2$

$a_3 = 2^{3-1} = 2^2 = 4$

$a_4 = 2^{4-1} = 2^3 = 8$

EXAMPLE 5

Given the general term: $a_n = -23 + 7n$, find the first four terms of the sequence

Solution:

$a_1 = -23 + 7(1) = -16$

$a_2 = -23 + 7(2) = -9$

$a_3 = -23 + 7(3) = -2$

$a_4 = -23 + 7(4) = 5$

EXAMPLE 6

Given the general term: $a_n = \dfrac{n+1}{2n+3}$, find the fifth and the tenth terms of the sequence

Solution:

$$a_5 = \frac{(5)+1}{2(5)+3} = \frac{6}{13}$$

$$a_{10} = \frac{(10)+1}{2(10)+3} = \frac{11}{23}$$

EVALUATING A SEQUENCE USING AN RECURSIVE FORMULA

A recursive formula defines each term of a sequence in relation to one or more of its preceding terms. Unlike an explicit formula, which allows you to jump directly to any term, a recursive formula requires you to know the previous term(s) to find the next one.

To evaluate a sequence using a recursive formula, follow these steps:

Identify the initial term(s). A recursive formula always provides one or more starting values, such as a_1 (and sometimes a_2 for second-order recursions).

Identify the recurrence relation. This is the rule that tells you how to generate the next term from the previous one(s). It is usually written in the form $a_n = f(a_{n-1})$

Apply the recurrence relation repeatedly. Starting from the initial term(s), use the rule to compute the next term, then use that result to compute the following term, and continue until you reach the desired term.

EXAMPLE 7

Given the recursive formula: $a_n = 2 \cdot a_{n-1}, a_1 = 4$, find the fourth term of the sequence

Solution:

$a_1 = 4$

$a_2 = 2 \cdot a_1 = 8$

$a_3 = 2 \cdot a_2 = 16$

$a_4 = 2 \cdot a_3 = 32$

EXAMPLE 8

Given the recursive formula: $a_n = a_{n-1} + n$, $a_1 = 9$, find the fourth term of the sequence

Solution:

$a_1 = 9$

$a_2 = a_1 + n = 9 + 2 = 11$

$a_3 = a_2 + n = 11 + 3 = 14$

$a_4 = a_4 + n = 14 + 4 = 18$

EXAMPLE 9

Given the recursive formula: $a_n = na_{n-1}$, $a_1 = 2$, find the fourth term of the sequence

Solution:

$a_1 = 2$

$a_2 = na_1 = 2(2) = 4$

$a_3 = na_2 = 3(4) = 12$

$a_4 = na_3 = 4(12) = 48$

DEFINITION ARITHMETIC SEQUENCE

A sequence is arithmetic when the difference between any two successive terms is constant.

The constant difference is called the common difference and is denoted by *d*.

EXAMPLE 10

Determine if the sequence $-30, -25, -20, -15, \ldots$ is arithmetic. If it is, find the common difference.

Solution:

Check if the difference between consecutive terms is constant.

Difference between term 2 and term 1: $-25 - (-30) = 5$

Difference between term 3 and term 2: $-20 - (-25) = 5$

Difference between term 4 and term 3: $-15 - (-20) = 5$

Since the difference between every pair of successive terms is the same ($d = 5$), the sequence is arithmetic.

EXAMPLE 11

Determine if the sequence $1, \frac{3}{2}, 2, \frac{7}{2}, \ldots$ is arithmetic. If it is, find the common difference.

Solution:

Check if the difference between consecutive terms is constant.

Difference between term 2 and term 1: $\frac{3}{2} - 1 = \frac{1}{2}$

Difference between term 3 and term 2: $2 - \frac{3}{2} = \frac{1}{2}$

Difference between term 4 and term 3: $\frac{7}{2} - 2 = \frac{3}{2}$

Since the difference between every pair of successive terms is NOT the same, the sequence is NOT arithmetic.

EXAMPLE 12

Determine if the sequence $a_n = \frac{6n+1}{4}$ is arithmetic. If it is, find the common difference.

Solution:

$a_1 = \frac{7}{4}, \quad a_2 = \frac{13}{4}, \; a_3 = \frac{19}{4}, \; a_4 = \frac{25}{4}$

Check if the difference between consecutive terms is constant.

Difference between term 2 and term 1: $\frac{13}{4} - \frac{7}{4} = \frac{6}{4} = \frac{3}{2}$

Difference between term 3 and term 2: $\frac{19}{4} - \frac{13}{4} = \frac{6}{4} = \frac{3}{2}$

Difference between term 4 and term 3: $\frac{25}{4} - \frac{19}{4} = \frac{6}{4} = \frac{3}{2}$

Since the difference between every pair of successive terms is the same $\left(d = \frac{3}{2} \right)$, the sequence is arithmetic.

NOTE:

RELATION WITH EQUATION OF A LINE

Arithmetic sequences are related to linear functions. For example, the arithmetic sequence $a_n = 2n - 1$ is related to the linear function $f(x) = 2x - 1$. The only difference is that $f(x)$ is defined for all real numbers *x*, while a_n is defined only for positive integers *n*.

ARITHMETIC SEQUENCE FORMULA

Explicit -General Term: $a_n = a_1 + (n-1)d$ - Recursive -Definition: $a_n = a_{n-1} + d$

In general, the arithmetic sequence $a_n = a_1 + (n-1)d$ can be rewritten as $a_n = dn + q$.
In this form we can see that the slope is the common difference *d*:
common difference $d = slope\ m$ of $f(x) = mx + q$

In Example 12 the sequence $a_n = \frac{6n+1}{4}$ could have been written as $a_n = \frac{3}{2}n + \frac{1}{4}$.

EXAMPLE 13

Given the first term $a_1 = 10$, the common difference $d = -3$ of an arithmetic sequence find the explicit formula, then find a_{21}

Solution:

Using $a_n = a_1 + (n-1)d$ we have:

$a_n = 10 + (n-1)(-3)$

Then

$a_{21} = 10 + (21-1)(-3) = 10 - 60 = -50$

EXAMPLE 14

Given the first term $a_1 = -\frac{13}{9}$, the common difference $d = \frac{5}{3}$ of an arithmetic sequence find the explicit formula, then find a_{33}

Solution:

Using $a_n = a_1 + (n-1)d$ we have:

$a_n = -\frac{13}{9} + (n-1)\left(\frac{5}{3}\right)$

Then

$a_{33} = -\frac{13}{9} + (33-1)\left(\frac{5}{3}\right) = -\frac{13}{9} + \frac{160}{3} = \frac{467}{9}$

EXAMPLE 15

Given the arithmetic sequence $a_n = -3n + 28$ find the recursive formula.

Solution:

Arithmetic sequence $a_n = -3n + 28$ is related to linear function $f(x) = -3x + 28$ with slope - 3.

Therefore the common difference is $d = -3$. Then $a_1 = -3(-1) + 28 = 25$. Therefore the recursive formula is $a_n = a_{n-1} - 3;\ a_1 = 25$

EXAMPLE 16

Given the arithmetic sequence $a_n = \frac{5}{3}n - \frac{32}{3}$ find the recursive formula.

Solution:

Arithmetic sequence $a_n = \frac{5}{3}n - \frac{32}{3}$ is related to linear function $f(x) = \frac{5}{3}x - \frac{32}{3}$ with slope $\frac{5}{3}$. Therefore the common difference is $d = \frac{5}{3}$. Then $a_1 = \frac{5}{3}(1) - \frac{32}{3} = -9$. Therefore the recursive formula is

$a_n = a_{n-1} + \frac{5}{3};\ a_1 = -9$

EXAMPLE 17

Given two terms $a_{10} = -282$ and $a_{33} = -972$ in an arithmetic sequence find a_{21}

Solution:

Method 1: System of equations

Using $a_n = a_1 + (n-1)d$ we have:

$a_{33} = a_1 + 32d$
$a_{10} = a_1 + 9d$ substitute the given values $-972 = a_1 + 32d$
$-282 = a_1 + 9d$

subtract the second equation from the first equation:

$(-972) - (-282) = (a_1 + 32d) - (a_1 + 9d)$

$-690 = 23d$

$d = -30$

Substitute back to find a_1:

$-282 = a_1 + 9(-30)$

$a_1 = -282 + 270 = -12$

The general term: $a_n = -12 + (n-1)(-30)$

So we have: $a_{21} = -12 + (20)(-30) = -612$

Method 2: Slope = common difference-equation of a line

Remember that common difference $d = slope\ m$ of $f(x) = mx + q$ and the general term is $a_n = dn + q$

We are given when $n = 10$ then $a_{10} = -282$ and when $n = 33$ then $a_{33} = -972$. It is like having two points: $(10, -282)$ and $(33, -972)$.

Find the slope = common difference $d = \frac{-972 - (-282)}{33 - 10} = \frac{-690}{23} = -30$

Using $a_{10} = -30(10) + q$ we can find q:

$-282 = -30(10) + q$

$q = 18$

Therefore the general term is: $a_n = -30n + 18 \rightarrow a_{21} = -30(21) + 18 = -630 + 18 = -612$

EXAMPLE 18

Given two terms $a_{12}=-\frac{16}{3}$ and $a_{36}=-\frac{64}{3}$ in an arithmetic sequence find a_{30}

Solution:

Method 1: System of equations

Using $a_n=a_1+(n-1)d$ we have:

$$\begin{aligned}a_{36}&=a_1+35d\\ a_{12}&=a_1+11d\end{aligned}$$ substitute the given values $$\begin{aligned}-\frac{64}{3}&=a_1+35d\\ -\frac{16}{3}&=a_1+11d\end{aligned}$$

subtract the second equation from the first equation:

$$\left(-\frac{64}{3}\right)-\left(-\frac{16}{3}\right)=(a_1+35d)-(a_1+11d)\ \rightarrow -\frac{48}{3}=24d\quad\rightarrow d=-\frac{16}{24}=-\frac{2}{3}$$

Substitute back to find a_1:

$$-\frac{16}{3}=a_1+11\left(-\frac{2}{3}\right)\ \rightarrow\ a_1=2$$

The general term: $a_n=2+(n-1)\left(-\frac{2}{3}\right)$

So we have: $a_{30}=2+(29)\left(-\frac{2}{3}\right)=-\frac{52}{3}$

Method 2: Slope = common difference-equation of a line

Remember that common difference $d=slope\ m$ of $f(x)=mx+q$ and the general term is $a_n=dn+q$

We are given when $n=12$ then $a_{12}=-\frac{16}{3}$ and when $n=36$ then $a_{36}=-\frac{64}{3}$. It is like having two points:

$\left(12,-\frac{16}{3}\right)$ and $\left(36,-\frac{64}{3}\right)$.

Find the slope = common difference $d=\frac{-\frac{64}{3}-\left(-\frac{16}{3}\right)}{36-12}=\frac{-16}{24}=-\frac{2}{3}$

Using $a_{12}=-\frac{2}{3}(12)+q$ we can find q:

$$-\frac{16}{3}=-8+q$$

$$q=\frac{8}{3}$$

Therefore the general term is: $a_n=-\frac{2}{3}n+\frac{8}{3}\rightarrow a_{30}=-\frac{2}{3}(30)+\frac{8}{3}=-20+\frac{8}{3}=-\frac{52}{3}$

EXAMPLE 19

The sum of three numbers in arithmetic sequence is $\frac{18}{5}$, and the product of the first and second is $\frac{24}{25}$.

Find the three numbers.

Solution:

Let the three terms of the arithmetic sequence be a, b, c

Because consecutive terms share a constant common difference d, we can express the first and third terms relative to the middle term b:

$a = b - d$ and $c = b + d$.

Adding the three terms: $(b-d)+b+(b+d) = 3b = \frac{18}{5} \rightarrow b = \frac{6}{5}$

Since the product of the first and second is $\frac{24}{25}$ we have:

$$(b-d)b = \frac{24}{25} \rightarrow \left(\frac{6}{5}-d\right)\frac{6}{5} = \frac{24}{25} \rightarrow \frac{36}{25} - \frac{6}{5}d = \frac{24}{25} \rightarrow d = \frac{2}{5}$$

Then: $a = \frac{6}{5} - \frac{2}{5} = \frac{4}{5}$ and $c = \frac{6}{5} + \frac{2}{5} = \frac{8}{5}$

EXAMPLE 20

The terms $\log_2 x, \log_4(x+3)$ and $\log_8(4x)$ form an arithmetic sequence in that specific order. Solve for all possible real values of x.

Solution:

Let the three terms of the arithmetic sequence be a, b, c and rewrite all terms using a common base of 2.

$$a = \log_2 x,\ b = \log_4(x+3) = \frac{1}{2}\log_2(x+3),\ c = \log_8(4x) = \frac{1}{3}\log_2(4x)$$

For three terms a, b and c to form an arithmetic sequence, the middle term must be the average of the outer terms $2b = a + c \rightarrow 2\left[\frac{1}{2}\log_2(x+3)\right] = \log_2 x + \frac{1}{3}\log_2(4x)$

Simplify the left side on the right side using the exponent rule:

$\log_2(x+3) = \log_2 x + \log_2(4x)^{\frac{1}{3}}$. Using the product rule: $\log_2(x+3) = \log_2\left[x(4x)^{\frac{1}{3}}\right]$

Having the same base: $x+3 = x(4x)^{\frac{1}{3}} \rightarrow (x+3)^3 = x^3(4x) \rightarrow x^3 + 9x^2 + 27x + 27 = 4x^4$

In standard polynomial form: $4x^4 - x^3 - 9x^2 - 27x - 27 = 0$

We test for rational roots using factors of the constant term 27 (like ± 1, ± 3). Let's test $x = 3$:

$4(3)^4 - (3)^3 - 9(3)^2 - 27(3) - 27 = 0 \rightarrow 4(81) - 27 - 81 - 81 - 27 = 0 \rightarrow 324 - 324 = 0$

So $x = 3$ is a valid solution. Factoring $(x-3)$ gives: $(x-3)(4x^3 + 11x^2 + 24x + 9) = 0$

For the terms to exist in the original logarithms, x must be greater than 0. Since all coefficients in the remaining cubic part $4x^3 + 11x^2 + 24x + 9$ are positive, it has no positive real roots. The only real solution to the equation is $x = 3$.

PRACTICE PROBLEMS

Find the next three terms in each sequence.

1) 1, 4, 9, 16, 25, ... 36, 49, 64

2) $-27,\ 9,\ -3,\ 1,\ -\frac{1}{3},\ \ldots$ $\frac{1}{9},\ -\frac{1}{27},\ \frac{1}{81}$

3) 1, 4, 16, 64, 256, ...
1024, 4096, 16384

4) −1, 8, 17, 26, 35, ...
44, 53, 62

Find the first four terms in each sequence.

5) $a_n = -\frac{243}{32}\cdot\left(\frac{2}{3}\right)^{n-1}$ $-\frac{243}{32},\ -\frac{81}{16},\ -\frac{27}{8},\ -\frac{9}{4}$

6) $a_n = 41 - 4n$
37, 33, 29, 25

7) $a_n = 4^{n-1}$
1, 4, 16, 64

8) $a_n = -13 - 5n$
−18, −23, −28, −33

Write the explicit formula for each sequence.

9) $5,\ \frac{10}{3},\ \frac{5}{2},\ 2,\ \frac{5}{3},\ \ldots$ $a_n = \frac{10}{n+1}$

10) $\frac{2}{3},\ \frac{4}{5},\ \frac{8}{7},\ \frac{16}{9},\ \frac{32}{11},\ \ldots$ $a_n = \frac{2^n}{2n+1}$

11) $\frac{3}{2},\ \frac{5}{4},\ \frac{7}{6},\ \frac{9}{8},\ \frac{11}{10},\ \ldots$ $a_n = \frac{2n+1}{2n}$

12) $2,\ 2,\ \frac{8}{3},\ 4,\ \frac{32}{5},\ \ldots$ $a_n = \frac{2^n}{n}$

Find the first four terms in each sequence.

13) $a_n = a_{n-1} + 1.8$ 6.6, 8.4, 10.2, 12
$a_1 = 6.6$

14) $a_n = \frac{2 + a_{n-1}}{2}$ −30, −14, −6, −2
$a_1 = -30$

15) $a_n = a_{n-1} + n$ 4, 6, 9, 13
$a_1 = 4$

16) $a_n = a_{n-1} + 20$ −39, −19, 1, 21
$a_1 = -39$

Write the recursive formula for each sequence.

17) −6, −4, −1, 3, 8, ... $a_n = a_{n-1} + n$
$a_1 = -6$

18) $2,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 0,\ \ldots$ $a_n = a_{n-1} - \frac{1}{2}$
$a_1 = 2$

19) 0, 4, 8, 12, 16, ... $a_n = a_{n-1} + 4$
$a_1 = 0$

20) 2, 4, 12, 48, 240, ... $a_n = na_{n-1}$
$a_1 = 2$

Determine if the sequence is arithmetic. If it is, find the common difference.

21) 26, 34, 42, 50, ... $d = 8$

22) −36, −34, −32, −30, ... $d = 2$

23) 18, 10, 6, 4, ...
Not arithmetic

24) 21, 26, 31, 36, ...
$d = 5$

Given a term in an arithmetic sequence and the common difference find the explicit formula.

25) $a_{24} = -203,\ d = -9$

Explicit: $a_n = 13 - 9n$

Recursive: $a_n = a_{n-1} - 9$, $a_1 = 4$

26) $a_{13} = 57,\ d = 3$

Explicit: $a_n = 18 + 3n$

Recursive: $a_n = a_{n-1} + 3$, $a_1 = 21$

27) $a_{10} = -75,\ d = -10$ Explicit: $a_n = 25 - 10n$

Recursive: $a_n = a_{n-1} - 10$, $a_1 = 15$

28) $a_{21} = 415,\ d = 20$ Explicit: $a_n = -5 + 20n$

Recursive: $a_n = a_{n-1} + 20$, $a_1 = 15$

Given the first term and the common difference of an arithmetic sequence find the term named in the problem and the explicit formula.

29) $a_1 = 12.3,\ d = 2.2$ $a_{33} = 82.7$

Find a_{33} Explicit: $a_n = 10.1 + 2.2n$

30) $a_1 = -6,\ d = -10$ $a_{22} = -216$

Find a_{22} Explicit: $a_n = 4 - 10n$

Given the explicit formula for an arithmetic sequence find the recursive formula.

31) $a_n = -\frac{29}{30} + \frac{5}{3}n$ $a_n = a_{n-1} + \frac{5}{3}$, $a_1 = \frac{7}{10}$

32) $a_n = -19 + 30n$ $a_n = a_{n-1} + 30$, $a_1 = 11$

Given the recursive formula for an arithmetic sequence find the explicit formula.

33) $a_n = a_{n-1} + \frac{4}{3}$ $a_n = -\frac{1}{6} + \frac{4}{3}n$

$a_1 = \frac{7}{6}$

34) $a_n = a_{n-1} - 2$

$a_1 = 24$

$a_n = 26 - 2n$

Given two terms in an arithmetic sequence find the term named in the problem.

35) $a_{18} = \frac{51}{2}$ and $a_{36} = \frac{105}{2}$ $a_{33} = 48$

Find a_{33}

36) $a_{15} = -\frac{338}{15}$ and $a_{35} = -\frac{838}{15}$ $a_{38} = -\frac{913}{15}$

Find a_{38}

37) $a_{16} = -32.5$ and $a_{33} = -63.1$ $a_{27} = -52.3$

Find a_{27}

38) $a_{17} = \frac{38}{3}$ and $a_{33} = \frac{70}{3}$ $a_{36} = \frac{76}{3}$

Find a_{36}

39) $a_{19} = -18.9$ and $a_{40} = -48.3$ $a_{36} = -42.7$

Find a_{36}

40) $a_{18} = -46$ and $a_{40} = -112$ $a_{21} = -55$

Find a_{21}

41) $a_{12} = 68$ and $a_{39} = 203$ $a_{25} = 133$

Find a_{25}

42) $a_{14} = \frac{194}{9}$ and $a_{39} = \frac{569}{9}$ $a_{31} = \frac{449}{9}$

Find a_{31}

43) The sum of three numbers in arithmetic sequence is $\frac{24}{5}$, and the product of the first and third is $\frac{36}{25}$.

Find the three numbers. $\frac{8-2\sqrt{7}}{5}, \frac{8}{5}, \frac{8+2\sqrt{7}}{5}$

44) The sum of three numbers in arithmetic sequence is $\frac{27}{4}$, and the product of the first and second is $\frac{45}{8}$.

Find the three numbers. $\frac{5}{2}, \frac{9}{4}, 2$

45) The roots of the cubic polynomial $x^3 - 15x^2 + kx - 105 = 0$ form an arithmetic sequence.

Find k and the three roots. $k = 71$; *roots* : $3, 5, 7$

4.2 Geometric Sequences

Topic: Definition, Recursive and Explicit Formula

Review

Given two terms in an arithmetic sequence find the term named in the problem.

1) $a_{17} = 97$ and $a_{39} = 185$
Find a_{34}

$a_{34} = 165$

2) $a_{19} = 349$ and $a_{35} = 669$
Find a_{28}

$a_{28} = 529$

Solve

3) $\log_3 (x^2 - 4) - \log_3 7 = 1$

$\{5, -5\}$

Write a polynomial function of least degree with integral coefficients that has the given zeros.

4) $-1 + \sqrt{2},\ 1 + \sqrt{7},\ 1 - \sqrt{7}$

$f(x) = x^4 - 11x^2 - 10x + 6$

GEOMETRIC SEQUENCE-DEFINITION

A sequence is geometric when the ratio (quotient) of every successive terms is constant

EXAMPLE 1

Determine if the sequence $1,3,7,15,...$ is geometric. If it is, find the common ratio.

Solution:

Let's examine the ratio of consecutive terms:

$\frac{3}{1} = 3,\ \frac{7}{3}....$

We can immediately stop: the ratio are not constant. Therefore the sequence is NOT geometric

EXAMPLE 2

Determine if the sequence $-3,15,-75,375,...$ is geometric. If it is, find the common ratio.

Solution:

Let's examine the ratio of consecutive terms:

$\frac{15}{-3} = -5;\ \frac{-75}{15} = -5;\ \frac{375}{-75} = -5$

Since the ratio $r = -5$ is constant, the sequence is geometric

GEOMETRIC SEQUENCE-FORMULA - RELATION WITH EXPONENTIAL FUNCTIONS

General Term (explicit): $u_n = u_1 r^{n-1}$

Recursive (Definition): $u_n = u_{n-1} r$

Just as arithmetic sequence and linear function are related, geometric sequences and exponential functions are also related. We can see it from the comparison of the sequence $u_n = 10\left(\frac{1}{2}\right)^{n-1}$ and the function $f(x) = 10\left(\frac{1}{2}\right)^x$

EXAMPLE 3

Given the explicit formula for geometric sequence $u_n = -2(-2)^{n-1}$ find the first four terms and the 8th term.

Solution:

$u_1 = -2(-2)^{1-1} = -2(1) = -2;\ u_2 = -2(-2)^{2-1} = -2(-2) = 4; u_3 = -2(-2)^{3-1} = -2(-2)^2 = -8;\ u_4 = -2(-2)^{4-1} = -2(-2)^3 = 16$

$u_8 = -2(-2)^{8-1} = -2(-2)^7 = -2(-128) = 256$

EXAMPLE 4

Given the explicit formula for geometric sequence $u_n = \frac{1}{3}(-3)^{n+1}$ find the first three terms and the 7th term.

Solution:

$u_1 = \frac{1}{3}(-3)^{1+1} = \frac{1}{3}(9) = 3;\ u_2 = \frac{1}{3}(-3)^{2+1} = \frac{1}{3}(-27) = -9;\ u_3 = \frac{1}{3}(-3)^{3+1} = \frac{1}{3}(81) = 27$

$u_7 = \frac{1}{3}(-3)^{7+1} = \frac{1}{3}(6561) = 2187$

EXAMPLE 5

Given two terms in the geometric sequence $u_3 = -48$ and $u_2 = -12$ find the recursive formula.

Solution:

Being a geometric sequence the ratio is constant, so we have $r = \frac{u_3}{u_2} = \frac{-48}{-12} = 4$

It follows that $u_1 = \frac{u_2}{r} = \frac{-12}{3} = -3$. The recursive formula is: $u_n = u_{n-1} \cdot 4;\quad u_1 = 3$

EXAMPLE 6

Given two terms in the geometric sequence $u_6 = -1280$ and $u_1 = -\frac{5}{4}$ find the recursive formula.

Solution:

Using the general formula $u_n = u_1 r^{n-1}$, we have $u_6 = u_1 r^5 \rightarrow r^5 = \frac{u_6}{u_1}$.

Substitute the value: $r^5 = \frac{-1280}{-\frac{5}{4}} = 1024 \rightarrow r = \sqrt[5]{1024} = 4$

The recursive formula is: $u_n = u_{n-1} \cdot 4; \quad u_1 = -\frac{5}{4}$

EXAMPLE 7

Given two terms in the geometric sequence $u_5 = 162$ and $u_2 = -6$ find u_{10}

Solution:

The difference between 5 and 2 is 3, it follows that $u_5 = u_2 r^3 \rightarrow r^3 = \frac{u_5}{u_2}$.

Substitute the value: $r^3 = \frac{162}{-6} = -27 \rightarrow r = \sqrt[3]{-27} = -3$

Then $u_{10} = u_5 r^5 = 162(-3)^5 = -39366$

EXAMPLE 8

The 3rd, 7th, and 14th terms of a non-constant arithmetic sequence form the first three consecutive terms of a geometric sequence. Find the common ratio of this geometric sequence.

Solution:

According to the problem we have $a_3 = u_1; \; a_7 = u_2; \; a_{14} = u_3$

Using the general formula for an arithmetic sequence: $a_n = a_1 + (n-1)d$ we have

$a_3 = a_1 + 2d = u_1; \; a_7 = a_1 + 6d = u_2; \quad a_{14} = a_1 + 13d = u_3$

For three terms u_1, u_2 and u_3 to form a geometric sequence, the ratio between consecutive terms must be equal: $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$. This means that $u^2{}_2 = u_1 \cdot u_3$, using the relation with the arithmetic sequence we have:

$(a_1 + 6d)^2 = (a_1 + 2d)(a_1 + 13d) \rightarrow a^2{}_1 + 12a_1 d + 36d^2 = a^2{}_1 + 15a_1 d + 26d^2$

Rearranging: $36d^2 - 26d^2 = 15a_1 d - 12a_1 d \rightarrow 10d^2 = 3a_1 d \rightarrow a_1 = \frac{10}{3}d$.

Substitute: $r = \frac{u_2}{u_1} = \frac{a_1 + 6d}{a_1 + 2d} = \frac{\left(\frac{10}{3}\right)d + 6d}{\left(\frac{10}{3}\right)d + 2d} = \frac{\left(\frac{10}{3}\right) + \frac{18}{3}}{\left(\frac{10}{3}\right) + \frac{6}{3}} = \frac{\frac{28}{3}}{\frac{16}{3}} = \frac{7}{4}$

EXAMPLE 9

Three numbers form a geometric sequence. If the base-2 exponents of these numbers form an arithmetic sequence with a common difference of 3, and the product of the three geometric terms is 4096, find the three terms of the geometric sequence.

Solution:

Let the terms be $u_1 = 2^{m-3}$, $u_2 = 2^m$ and $u_3 = 2^{m+3}$

Multiply $\left(2^{m-3}\right)\left(2^m\right)\left(2^{m+3}\right) = 4096 \rightarrow 2^{3m} = 2^{12} \rightarrow m = 4$

Substitute

$u_1 = 2$, $u_2 = 2^4 = 16$ and $u_3 = 2^7 = 128$

EXAMPLE 10

Let x, y and z be three consecutive terms of a geometric sequence with common ratio r. If $x, 2y$ and $3z$ form an arithmetic sequence in that order, find all possible real values of r.

Solution:

Geometric condition: $x, y = xr$ and $z = xr^2$

Arithmetic condition: $x, 2y, 3z$

The three terms $x, 2y, 3z$ form an arithmetic sequence if the middle term must be the average of the outer terms, which means: $2y = \dfrac{x + 3z}{2} \rightarrow 4y = x + 3z$. Since $y = xr$ and $z = xr^2$ we have:

$4(xr) = x + 3\left(xr^2\right)$

Set equal to zero

$3xr^2 - 4xr + x = 0$

$x\left(3r^2 - 4r + 1\right) = 0$ however $x \neq 0$ so factor the quadratic:

$(3r - 1)(r - 1) = 0$

So $r = 1$ (constant sequence where $x = y = z$) then $r = \dfrac{1}{3}$

EXAMPLE 11

In a geometric sequence of positive real numbers, the product of the 2nd and 6th terms is 16, and the ratio of the 7th term to the 3rd term is 9. Find the 1st term and the common ratio r.

Solution:

Given: $u_2 \cdot u_6 = 16$ and $\dfrac{u_7}{u_3} = 9$. Using the formula $u_n = u_1 r^{n-1}$ we have:

$(u_1 r) \cdot \left(u_1 r^5\right) = 16$ and $\dfrac{u_1 r^6}{u_1 r^2} = 9 \rightarrow \left(u_1\right)^2 r^6 = 16$ and $r^4 = 9 \rightarrow r = \sqrt{3}$

Substitute in $\left(u_1\right)^2 r^6 = 16$ and solving for u_1:

$\left(u_1\right)^2 \left(\sqrt{3}\right)^6 = 16 \rightarrow \left(u_1\right)^2 27 = 16 \rightarrow u_1 = \dfrac{4}{3\sqrt{3}}$

PRACTICE PROBLEMS

Determine if the sequence is geometric. If it is, find the common ratio.

1) 2, 6, 18, 54, ...

$r = 3$

2) $-1, -2, -4, -8, \ldots$

$r = 2$

3) 2, 8, 32, 128, ...

$r = 4$

4) $3, -9, 27, -81, \ldots$

$r = -3$

5) $-8, -6, -3, 1, \ldots$

Not geometric

6) $-2, -8, -32, -128, \ldots$

$r = 4$

7) $-1, -5, -25, -125, \ldots$

$r = 5$

8) 4, 8, 16, 32, ...

$r = 2$

Given the explicit formula for a geometric sequence find the first five terms and the 8th term.

9) $a_n = 3 \cdot \left(\frac{1}{5}\right)^{n-1}$

First Five Terms: $3, \frac{3}{5}, \frac{3}{25}, \frac{3}{125}, \frac{3}{625}$

$a_8 = \frac{3}{78125}$

10) $a_n = 3 \cdot 5^{n-1}$

First Five Terms: 3, 15, 75, 375, 1875

$a_8 = 234375$

11) $a_n = \frac{3}{4} \cdot \left(-\frac{1}{3}\right)^{n-1}$

First Five Terms: $\frac{3}{4}, -\frac{1}{4}, \frac{1}{12}, -\frac{1}{36}, \frac{1}{108}$

$a_8 = -\frac{1}{2916}$

12) $a_n = 4 \cdot (-5)^{n-1}$

First Five Terms: 4, −20, 100, −500, 2500

$a_8 = -312500$

13) $a_n = 0.6 \cdot 5^{n-1}$

First Five Terms: 0.6, 3, 15, 75, 375

$a_8 = 46875$

14) $a_n = -2 \cdot 5^{n-1}$

First Five Terms: −2, −10, −50, −250, −1250

$a_8 = -156250$

15) $a_n = -3 \cdot 5^{n-1}$

First Five Terms: −3, −15, −75, −375, −1875

$a_8 = -234375$

16) $a_n = (-4)^{n-1}$

First Five Terms: 1, −4, 16, −64, 256

$a_8 = -16384$

Given two terms in a geometric sequence find the recursive formula.

17) $a_6 = -96$ and $a_1 = -3$

$a_n = a_{n-1} \cdot 2$

$a_1 = -3$

18) $a_6 = 7776$ and $a_5 = -1296$

$a_n = a_{n-1} \cdot -6$

$a_1 = -1$

19) $a_3 = -36$ and $a_2 = -6$

$a_n = a_{n-1} \cdot 6$

$a_1 = -1$

20) $a_6 = -15552$ and $a_1 = -2$

$a_n = a_{n-1} \cdot 6$

$a_1 = -2$

21) $a_4 = -256$ and $a_1 = 4$

$a_n = a_{n-1} \cdot -4$

$a_1 = 4$

22) $a_3 = 3$ and $a_6 = -\frac{24}{125}$ $a_n = a_{n-1} \cdot -\frac{2}{5}$

$a_1 = \frac{75}{4}$

23) $a_4 = \frac{1}{216}$ and $a_5 = \frac{1}{1296}$ $a_n = a_{n-1} \cdot \frac{1}{6}$

$a_1 = 1$

24) $a_3 = 144$ and $a_4 = -864$

$a_n = a_{n-1} \cdot -6$

$a_1 = 4$

Given two terms in a geometric sequence find the term named in the problem.

25) $a_5 = \frac{1}{81}$ and $a_2 = \frac{1}{3}$ $a_9 = \frac{1}{6561}$

Find a_9

26) $a_1 = -20$ and $a_6 = -\frac{5}{8}$ $a_9 = -\frac{5}{64}$

Find a_9

27) $a_1 = 2.5$ and $a_2 = -10$

Find a_{10}

$a_{10} = -655360$

28) $a_4 = -12$ and $a_1 = -1.5$

Find a_9

$a_9 = -384$

29) $a_6 = -1024$ and $a_1 = -1$

Find a_{10}

$a_{10} = -262144$

30) $a_4 = 32$ and $a_1 = -4$

Find a_{10}

$a_{10} = 2048$

31) $a_1 = 0.75$ and $a_4 = 48$

Find a_{10}

$a_{10} = 196608$

32) $a_2 = -12$ and $a_3 = 36$

Find a_9

$a_9 = 26244$

33) In a non-constant arithmetic sequence, the 2nd, 5th, and 11th terms form the first three consecutive terms of a geometric sequence. Find the common ratio of that geometric sequence. *r = 2*

34) In a non-constant arithmetic sequence, the 4th, 9th, and 19th terms form the first three consecutive terms of a geometric sequence. Find the common ratio. *r = 2*

35) Three numbers form a geometric sequence. If the base-3 logarithms of these numbers form an arithmetic sequence with common difference 2, and the product of the three geometric terms is 729, find the three terms. *1, 9 and 81*

36) Three numbers form a geometric sequence. If the base-5 logarithms of these numbers form an arithmetic sequence with common difference 1, and the product of the three geometric terms is 125, find the three terms. *1, 5 and 25*

37) Let x, y, z be three consecutive terms of a geometric sequence with common ratio r. If $x, 3y, 5z$ form an arithmetic sequence in that order, find all possible real values of r. $r = \frac{1}{5}$

38) Let x, y, z be three consecutive terms of a geometric sequence with common ratio r. If $2x, 4y, 7z$ form an arithmetic sequence in that order, find all possible real values of r.

$r = \frac{4+\sqrt{2}}{7}, \frac{4-\sqrt{2}}{7}$

39) In a geometric sequence of positive real numbers, the product of the 3rd and 7th terms is 36, and the ratio of the 8th term to the 4th term is 4. Find the 1st term and the common ratio r.

$r = \sqrt{2}, u_1 = \frac{3}{2}$

40) In a geometric sequence of positive real numbers, the product of the 1st and 4th terms is 27, and the ratio of the 6th term to the 3rd term is 27. Find the 1st term and the common ratio r. $r = 3, u_1 = 1$

4.3 ARITHMETIC SERIES

Topic: Sigma Notation, Arithmetic Series

Review

Given two terms in an arithmetic sequence find the term named in the problem.

1) $a_{17} = -\frac{28}{3}$ and $a_{30} = -\frac{95}{6}$ $a_{27} = -\frac{43}{3}$

Find a_{27}

Given two terms in a geometric sequence find the term named in the problem.

2) $a_1 = 4$ and $a_4 = 108$ $a_{10} = 78732$

Find a_{10}

Solve

3) $25^{1-n} \cdot 125^{-3n+3} = 625$ $\left\{\frac{7}{11}\right\}$

Find the inverse

4) $f(x) = \sqrt[3]{x+1} - 1$

$f^{-1}(x) = (x+1)^3 - 1$

A **series** is simply what you get when you take a list of numbers (a sequence) and add all the terms together. While a sequence lists items side-by-side $(2,4,6,8)$, a series adds them up $(2+4+6+8)$.

Sigma notation $\sum$ is the language built specifically to express series. It bundles the starting point, the stopping point, and the underlying pattern into one clear symbol, letting you represent both finite sums and infinite series. Series have vast applications across real-world fields:

Signal Processing and Audio (Fourier Series): Representing complex sound waves or images by summing basic sine and cosine waves—the foundation of MP3 and JPEG compression.

Calculus (Taylor Series): Expressing complex functions (like sines, logarithms, or exponentials) as simple polynomial series so calculators and computers can compute them rapidly.

Financial Modeling (Geometric Series): Calculating the total present value of lifetime annuity payouts or multi-year investment cash flows.

Physics and Quantum Mechanics: Summing infinite potential states to calculate total energy, wave behaviors, or particle probabilities.

SIGMA NOTATION - DEFINITION

Sigma notation or Summation notation is a concise and powerful way to represent the sum of a sequence of terms. The Greek letter $\sum$ (sigma) is used to denote summation.

Given a sequence $a_1, a_2, a_3, ..., a_n$, their sum can be written in sigma notation as:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$$

where

$\sum$ is the summation symbol

$i = 1$ is the index of summation (starting value)

n is the upper limit of summation (ending value)

a_i is the general term of the series (defines each term in the sum based on the index)

EXAMPLE 1

Rewrite as a sum. $\sum_{m=1}^{4} m^2$

Solution:

$$\sum_{m=1}^{4} m^2 = 1^2 + 2^2 + 3^2 + 4^2$$

EXAMPLE 2

Rewrite as a sum. $\sum_{i=1}^{6} \left(20 - i^2\right)$

Solution:

$$\sum_{i=1}^{6} \left(20 - i^2\right) = \left(20 - 1^2\right) + \left(20 - 2^2\right) + \left(20 - 3^2\right) + \left(20 - 4^2\right) + \left(20 - 5^2\right) + \left(20 - 6^2\right)$$

EXAMPLE 3

Rewrite using sigma notation: 101+102 + 103 + 104 + 105

Solution:

$$101+102 + 103 + 104 + 105 = \sum_{i=1}^{5} (i + 100)$$

EXAMPLE 4

Rewrite using sigma notation: 4 + 9 + 16 +25 + 36

Solution:

$$4 + 9 + 16 + 25 + 36 = \sum_{n=2}^{6} n^2$$

EXAMPLE 5

Tell whether or not the pair of series is the same: $\sum_{k=1}^{8}\left(\frac{1}{k}+3\right)=\sum_{i=3}^{11}\frac{1}{i}$

Solution:

Expand: $\sum_{k=1}^{8}\left(\frac{1}{k}+3\right)=\left(\frac{1}{1}+3\right)+\left(\frac{1}{2}+3\right)+\left(\frac{1}{3}+3\right)+\left(\frac{1}{4}+3\right)+\left(\frac{1}{5}+3\right)+\left(\frac{1}{6}+3\right)+\left(\frac{1}{7}+3\right)+\left(\frac{1}{8}+3\right)=$

$$=4+\frac{7}{2}+\frac{10}{3}+\frac{13}{4}+\frac{16}{5}+\frac{19}{6}+\frac{22}{7}+\frac{25}{8}$$

Expand: $\sum_{i=3}^{11}\frac{1}{i}=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}$

Values are entirely different. Therefore, the pair of series is not the same.

EXAMPLE 6

Tell whether or not the pair of series is the same: $\sum_{k=0}^{4}\left(\frac{1}{2^k}\right)=\sum_{i=3}^{7}\frac{1}{2^{i-3}}$

Solution:

Expand: $\sum_{k=0}^{4}\left(\frac{1}{2^k}\right)=\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}$

Expand: $\sum_{i=3}^{7}\frac{1}{2^{i-3}}=\frac{1}{2^{3-3=0}}+\frac{1}{2^{4-3=1}}+\frac{1}{2^{5-3=2}}+\frac{1}{2^{6-3=3}}+\frac{1}{2^{7-3=4}}$

Same values are entirely different. Therefore, the pair of series is the same.

NOTE:

This technique is called an *index shift*. To shift the starting index **up** by $3\left(\text{from } k=0 \text{ to } i=3\right)$, you compensate by subtracting 3 inside the formula $\left(2^k=2^{i-3}\right)$

SIGMA PROPERTIES

Addition:

$$\sum_{i=1}^{n}(x+y)=\sum_{i=1}^{n}x+\sum_{i=1}^{n}y$$

Subtraction:

$$\sum_{i=1}^{n}(x-y)=\sum_{i=1}^{n}x-\sum_{i=1}^{n}y$$

Sigma notation lacks the multiplication/division property:

$$\sum_{i=1}^{n}xy\neq\sum_{i=1}^{n}x\sum_{i=1}^{n}y$$

SIGMA FORMULAS

1. $\sum_{i=1}^{n}k=kn$
2. $\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n}i^3=\frac{n^2(n+1)^2}{4}$

EXAMPLE 7

Evaluate $\sum_{m=1}^{12} 4m$

Solution:

Since 4 is a constant multiplier, it can be factored out of the summation: $\sum_{m=1}^{12} 4m = 4\sum_{m=1}^{12} m$

Use the formula for the sum of the first n positive integers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Evaluate with $n = 12$

we have: $\sum_{m=1}^{12} 4m = 4\sum_{m=1}^{12} m = 4\left[\frac{12(12+1)}{2}\right] = 2\left[12(13)\right] = 312$

EXAMPLE 8

Evaluate $\sum_{i=1}^{14} (3i-2)$

Solution:

First apply the subtraction property: $\sum_{i=1}^{14} (3i-2) = \sum_{i=1}^{14} 3i - \sum_{i=1}^{14} 2$

Since 3 is a constant multiplier, it can be factored out of the summation:

$\sum_{i=1}^{14} (3i-2) = 3\sum_{i=1}^{14} i - \sum_{i=1}^{14} 2$

Use the formula for the sum of the first n positive integers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

and the constant formula: $\sum_{i=1}^{n} k = kn$

Evaluate with $n = 14$

we have: $\sum_{i=1}^{14} (3i-2) = 3\sum_{i=1}^{14} i - \sum_{i=1}^{14} 2 = 3\left[\frac{14(15)}{2}\right] - 2(14) = 315 - 28 = 287$

EXAMPLE 9

Evaluate $\sum_{m=5}^{9} m(m+1)$

Solution:

First rewrite: $\sum_{m=5}^{9} m(m+1) = \sum_{m=5}^{9} (m^2 + m)$

Now we can use *index shift*. Let $m = (i+4)$. So when $i = 1$ then $m = 5$ and when $i = 5$ then $m = 9$

$$\sum_{m=5}^{9} m(m+1) = \sum_{m=5}^{9} \left(m^2+m\right) = \sum_{i=1}^{5} \left[(i+4)^2+(i+4)\right]$$

Expand and simplify: $\sum_{i=1}^{5} \left[(i+4)^2+(i+4)\right] = \sum_{i=1}^{5} \left[\left(i^2+8i+16\right)+(i+4)\right] = \sum_{i=1}^{5} \left(i^2+9i+20\right)$

Apply Sigma properties:

$$\sum_{i=1}^{5} \left(i^2+9i+20\right) = \sum_{i=1}^{5} i^2 + 9\sum_{i=1}^{5} i + \sum_{i=1}^{5} 20$$

Use the formula for the sum of the first n squared, the n positive integers and the constant.

Evaluate with $n = 5$

We have:

$$\sum_{m=5}^{9} m(m+1) = \sum_{i=1}^{5} \left(i^2+9i+20\right) = \left[\frac{5(5+1)(2(5)+1)}{6}\right] + 9\left[\frac{5(6)}{2}\right] + 20(5) =$$

$$= \left[\frac{5(6)(10+1)}{6}\right] + 9\left[\frac{30}{2}\right] + 100 = 55+135+100 = 290$$

SUM OF AN ARITHMETIC SERIES

When evaluating the sum of an arithmetic sequence, we don't always need to rely on standard sigma formulas or term-by-term addition. Instead, we can use a specific formula designed just for arithmetic series.

The sum of the first n terms of an arithmetic series is: $S_n = \frac{n(a_1+a_n)}{2}$ or $S_n = \frac{n}{2}\left[2a_1+(n-1)d\right]$

Note:
any linear expression inside a sigma notation, like $2k+1$, always generates an arithmetic sequence.

EXAMPLE 10

Evaluate $\sum_{i=1}^{8} (8i-12)$

Solution:

It is an arithmetic sequence with $n = 8$, $a_1 = 8(1)-12 = -4$ and $a_8 = 8(8)-12 = 52$.

Apply $S_n = \frac{n(a_1+a_n)}{2}$ $\rightarrow$ $S_8 = \frac{8(-4+52)}{2} = 192$

EXAMPLE 11

Evaluate $\sum_{i=1}^{9} (6-10i)$

Solution:

It is an arithmetic sequence with $n = 9$, $a_1 = 6-10 = -4$ and $a_9 = 6-90 = -84$.

Apply $S_n = \dfrac{n(a_1 + a_n)}{2} \quad \rightarrow \quad S_9 = \dfrac{9(-4-84)}{2} = -396$

EXAMPLE 12

Determine the number of terms n in $\sum_{k=1}^{n}(3k+7) = 770$.

Solution:

Since $3k + 7$ is a linear expression, the terms form an arithmetic sequence.

Evaluate the first term:

$a_1 = 3(1) + 7 = 10$

Evaluate the nth term:

$a_n = 3(n) + 7 = 3n + 7$

Substitute in : $S_n = \dfrac{n(a_1 + a_n)}{2}$

$770 = \dfrac{n(10 + 3n + 7)}{2} \quad \rightarrow 770 = \dfrac{n(3n+17)}{2} \quad \rightarrow 1540 = 3n^2 + 17n \quad \rightarrow 3n^2 + 17n - 1540 = 0$

Factorize or use quadratic formula:

$(3n + 77)(n - 20) = 0$

$n = 20, \quad n = -\dfrac{77}{3}(\text{not valid; } n \text{ must be a positive integer})$

EXAMPLE 13

Determine the number of terms n in $\sum_{i=1}^{n}\left(\dfrac{17}{21} + \dfrac{i}{3}\right) = 15$.

Solution:

Since $\left(\dfrac{17}{21} + \dfrac{i}{3}\right)$ is a linear expression, the terms form an arithmetic sequence.

Evaluate the first term:

$a_1 = \left(\dfrac{17}{21} + \dfrac{1}{3}\right) = \dfrac{17}{21} + \dfrac{7}{21} = \dfrac{24}{21} = \dfrac{8}{7}$

Evaluate the nth term:

$a_n = \dfrac{17}{21} + \dfrac{n}{3}$

Substitute in : $S_n = \dfrac{n(a_1 + a_n)}{2}$

$15 = \dfrac{n\left(\dfrac{8}{7} + \dfrac{17}{21} + \dfrac{n}{3}\right)}{2} \quad \rightarrow 30 = \dfrac{41}{21}n + \dfrac{n^2}{3} \quad \rightarrow 7n^2 + 41n - 630 = 0$

Factorize:

$(7n+90)(n-7)=0$

$n=7,\ \ n=-\dfrac{90}{7}(\text{not valid; } n \text{ must be a positive integer})$

EXAMPLE 14

Determine the number of terms n in $\sum_{k=4}^{n}(5k-2)=1428$.

Solution:

Since $(5k-2)$ is a linear expression, the terms form an arithmetic sequence.

However, the lower bound does not start at 1. We need to make an *index shift*.

Let $k=i+3\rightarrow i=k-3$.

Lower limit: when $k=4\rightarrow i=4-3=1$

Upper limit: when $k=n\rightarrow i=n-3$

So $\sum_{k=4}^{n}(5k-2)=\sum_{i=1}^{n-3}(5(i+3)-2)=\sum_{i=1}^{n-3}(5i+13)=1428$

Evaluate the first term:

$a_1=5+13=18$

Evaluate the *last* term:

$a_{n-3}=5(n-3)+13=5n-2$

Substitute in : $S_n=\dfrac{n(a_1+a_n)}{2}$

$1428=\dfrac{(n-3)(18+5n+2)}{2}\qquad\rightarrow 2856=(n-3)(5n+16)\qquad\rightarrow 5n^2+n-2904=0$

Factorize:

$(5n+121)(n-24)=0$

$n=24,\ \ n=-\dfrac{121}{5}(\text{not valid; } n \text{ must be a positive integer})$

EXAMPLE 15

Consider the sequence $a_k=4k+1$. An arithmetic series S is formed by taking every 3rd term starting from a_2 up to a_{59} : $S=a_2+a_5+a_8+\ldots+a_{59}$. Calculate the total sum S.

Solution:

Evaluate the first term (a_2) : $4(2)+1=9$

Evaluate the *last* term (a_{59}): $4(59)+1=237$

To determine the number of terms look at the indices of these terms $a_2+a_5+a_8+\ldots+a_{59}$

these index numbers form their own small arithmetic sequence

- first index: 2
- common difference: 3
- last index: 59

Remember the formula for the *n*th term of an arithmetic sequence: $a_n = a_1 + (n-1)d$

Applying this directly to our indices:

$$\text{Last Index} = \text{First Index} + (n-1)(\text{Index Difference})$$

$$59 = 2 + (n-1)3$$

Solving for *n*:

$$57 = 3n - 3$$

$$n = 20$$

Evaluate the sum:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$S = \frac{20}{2}(9 + 237) = 10(246) = 2460$$

EXAMPLE 16

The sum of the first n terms of an arithmetic series is given by S_n. If $S_{10} = 150$ and $S_{20} = 500$.

Find the first term a_1 and common difference d.

Determine the smallest integer value of n such that $S_n > 2016$

Solution:

Using $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \rightarrow \quad S_{10} = \frac{10}{2}\left[2a_1 + 9d\right] = 150 \rightarrow 2a_1 + 9d = 30 \rightarrow$

$$S_{20} = \frac{20}{2}\left[2a_1 + 19d\right] = 500 \rightarrow 2a_1 + 19d = 50$$

Subtract the two equation to eliminate a_1:

$10d = 20 \rightarrow d = 2$

Solve for a_1: $\quad 2a_1 + 9(2) = 30 \rightarrow a_1 = 6$

Solving for n such that $S_n > 2016$ using $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$:

$$\frac{n}{2}\left[2(6) + (n-1)(2)\right] > 2016$$

$$n\left[12 + 2n - 2\right] > 4032$$

$$2n^2 + 10n - 4032 > 0$$

$$n^2 + 5n - 2016 > 0$$

$$(n-42)(n+48) > 0$$

For S_n to strictly exceed 2016, we need $n > 42$. Thus, the smallest integer value is $n > 43$.

PRACTICE PROBLEMS

Rewrite each series as a sum.

1) $\sum_{k=1}^{5}(100-k^2)$

$99+96+91+84+75$

2) $\sum_{a=1}^{6}(100-a^2)$

$99+96+91+84+75+64$

3) $\sum_{m=1}^{6}(5m^2-2)$

$3+18+43+78+123+178$

4) $\sum_{m=1}^{4}m(m+2)$

$3+8+15+24$

5) $\sum_{k=1}^{5}(5k^2+1)$

$6+21+46+81+126$

6) $\sum_{a=1}^{6}3^a$

$3+9+27+81+243+729$

Rewrite using sigma notation.

7) $0+1+2+3+4$ $\sum_{n=0}^{4}n$

8) $0+1+2+3$ $\sum_{n=0}^{3}n$

9) $305+306+307+308+309+310$ $\sum_{n=5}^{10}(n+300)$

10) $5+6+7+8+9+10$ $\sum_{k=5}^{10}k$

11) $4+8+12+16+20+24$ $\sum_{a=1}^{6}4a$

12) $1+2+3+4+5+6$ $\sum_{k=1}^{6}k$

13) Tell whether or not the pair of series is the same: $\sum_{k=0}^{4}\frac{3k+3}{k+6}=\sum_{j=1}^{3}\frac{3i}{i+5}$

different

14) Tell whether or not the pair of series is the same: $\sum_{k=0}^{\infty}\frac{k+1}{3^k}=\sum_{n=1}^{\infty}\frac{n}{3^{n-1}}$

same

15) Show that $\sum_{n=1}^{4}n\log 5=\log 5^{10}$

16) Write $\log 1\cdot2\cdot3\cdot4\cdot5\cdot6$ using sigma summation $\sum_{n=1}^{6}\log n$

Evaluate each series.

17) $\sum_{m=5}^{11}m^2$ 476

18) $\sum_{k=5}^{9}(3k^2+1)$ 770

19) $\sum_{n=5}^{9}n(n-2)$ 185

20) $\sum_{a=1}^{6}(40-a^2)$ 149

21) $\sum_{n=5}^{10}(500-n)$ 2955

22) $\sum_{a=1}^{7}5a$ 140

23) $\sum_{k=4}^{9}k$ 39

24) $\sum_{m=1}^{6}(500-m)$ 2979

Evaluate the related series of each sequence.

25) 37, 44, 51, 58, 65
255

26) −14, −20, −26, −32, −38, −44
−174

27) 9, 16, 23, 30, 37, 44
159

28) 33, 40, 47, 54, 61, 68, 75
378

29) 16, 22, 28, 34
100

30) 19, 24, 29, 34, 39, 44, 49
238

Evaluate each series described.

31) $\sum_{k=1}^{10}(4-9k)$ −455

32) $\sum_{n=1}^{45}(7n-6)$ 6975

33) $\sum_{m=1}^{12}(3m+2)$ 258

34) $\sum_{i=1}^{12}(10i-10)$ 660

35) $\sum_{i=1}^{45}(6i-2)$ 6120

36) $\sum_{n=1}^{6}(10n-5)$ 180

37) $\sum_{n=1}^{12}(10n-2)$ 756

38) $\sum_{k=1}^{5}7k$ 105

Determine the number of terms *n* in each series.

39) $\sum_{i=1}^{n}(3i-11)=195$ 15

40) $\sum_{i=1}^{n}(-2.3i-0.6)=-51.9$ 6

41) $\sum_{i=1}^{n}(2i-4)=10$ 5

42) $\sum_{k=1}^{n}(4-6k)=-102$ 6

43) $\sum_{m=1}^{n}(10m-8)=470$ 10

44) $\sum_{k=1}^{n}(-2k-7)=-240$ 12

45) $\sum_{k=1}^{n}(6k-9)=429$ 13

46) $\sum_{m=1}^{n}(1.5m-6.3)=54.6$ 13

47) Determine the number of terms *n* in $\sum_{k=4}^{n}(4k+3)=1575$. *n = 30*

48) Consider the sequence $a_k=5k-4$. Find the sum $S=a_3+a_6+a_9+\ldots+a_{57}$ *S = 2774*

49) The sum of the first *n* terms of an arithmetic series is given by S_n. If $S_{12}=216$ and $S_{24}=720$. Find the first term a_1 and common difference *d* and determine the smallest integer value of *n* such that $S_n>3000$. $a_1=7,\ d=2,\ n=52$

50) An arithmetic sequence has first term 12 and common difference 5. Find the smallest value of *n* such that $S_n>2500$ *n = 30*

4.4 GEOMETRIC SERIES

Topic: Sigma Notation, Finite and Infinite Series

Review

Given two terms in a geometric sequence find the term named in the problem.

1) $a_6 = 64$ and $a_1 = 2$

Find a_9

$a_9 = 512$

Evaluate

2) $\sum_{k=2}^{7} (50 - k)$

273

Determine the number of terms *n*

3) $\sum_{m=1}^{n} (3m - 10) = -5$

5

Solve each equation for $0 \le \theta < 2\pi$.

4) $\frac{-10 - \sqrt{3}}{2} = -5 - \frac{3}{2} \cdot \tan(\theta + \pi)$

$\left\{\frac{\pi}{6}, \frac{7\pi}{6}\right\}$

A geometric series is a specific type of series where each term after the first is found by multiplying the previous one by a constant value called the *common ratio*. Because each new term grows or shrinks by this same multiplying factor, geometric series have unique properties that allow us to easily calculate their sums and even sum an infinite number of terms if that ratio is small enough.

SUM OF A GEOMETRIC SERIES

The sum of the first *n* terms of a geometric series is: $S_n = \frac{u_1(1 - r^n)}{1 - r}, \; r \neq 1$

EXAMPLE 1

Evaluate $\sum_{m=1}^{7}(-2)^{m-1}$

Solution:

Apply $S_n = \dfrac{u_1\left(1-r^n\right)}{1-r}$

with $u_1 = (-2)^0 = 1$, $r = (-2)$ so we have

$$S_7 = \frac{1\left(1-(-2)^7\right)}{1-(-2)} = \frac{(1+128)}{3} = 43$$

EXAMPLE 2

Evaluate the related series of sequence. $4, 20, 100, 500, 2500$

Solution:

We have a geometric series with $u_1 = 4$, $r = 5$, $n = 5$, so apply $S_n = \dfrac{u_1\left(1-r^n\right)}{1-r}$:

$$S_5 = \frac{4\left(1-5^5\right)}{1-5} = -(1-3125) = 3124$$

EXAMPLE 3

Evaluate the geometric series with $u_1 = -2,\ u_n = 4374, r = -3$

Solution:

We need to find the number of terms n, using geometric sequence formula:

$u_n = u_1 r^{n-1}$

$$4374 = (-2)(-3)^{n-1} \rightarrow -2187 = (-3)^{n-1} \rightarrow (-3)^7 = (-3)^{n-1} \rightarrow n = 8$$

Apply $S_n = \dfrac{u_1\left(1-r^n\right)}{1-r}$:

$$S_8 = \frac{-2\left(1-(-3)^8\right)}{1-(-3)} = -\frac{(1-6561)}{2} = \frac{6560}{2} = 3280$$

EXAMPLE 4

Evaluate the geometric series with $-1-4-16-64\ldots n = 7$

Solution:

Apply $S_n = \dfrac{u_1\left(1-r^n\right)}{1-r}$ $\quad S_7 = \dfrac{-1\left(1-(4)^7\right)}{1-(4)} = \dfrac{(1-16384)}{3} = \dfrac{-16383}{3} = -5461$

EXAMPLE 5

Determine the number of terms n in $\sum_{k=1}^{n}\left(\frac{3}{5}\right)^{k-1}=\frac{1441}{625}$

Solution:

We have $u_1=1,\ r=\frac{3}{5}$

Use $S_n=\frac{u_1\left(1-r^n\right)}{1-r}$ and substitute

$$\frac{1441}{625}=\frac{1\left(1-\left(\frac{3}{5}\right)^n\right)}{1-\left(\frac{3}{5}\right)}$$

Solving for n:

$$\frac{1441}{625}=\frac{1-\left(\frac{3}{5}\right)^n}{\frac{2}{5}}\ \rightarrow\ \left(\frac{2}{5}\right)\frac{1441}{625}=1-\left(\frac{3}{5}\right)^n\ \rightarrow\ \frac{2882}{3125}=1-\left(\frac{3}{5}\right)^n$$

$$\rightarrow\ \frac{2882}{3125}-1=-\left(\frac{3}{5}\right)^n\ \rightarrow\ \frac{3125-2882}{3125}=-\left(\frac{3}{5}\right)^n\rightarrow\frac{243}{3125}=-\left(\frac{3}{5}\right)^n$$

$$\left(\frac{3}{5}\right)^5=\left(\frac{3}{5}\right)^n\rightarrow n=5$$

EXAMPLE 6

Determine the number of terms n in $\sum_{m=1}^{n}2(-6)^{m-1}=79982$

Solution:

We have $u_1=2,\ r=-6$

Use $S_n=\frac{u_1\left(1-r^n\right)}{1-r}$ and substitute

$$\frac{2\left(1-(-6)^n\right)}{1-(-6)}=79982$$

Solving for n:

$$\frac{2\left(1-(-6)^n\right)}{7}=79982\ \rightarrow\left(1-(-6)^n\right)=79982\left(\frac{7}{2}\right)\rightarrow 1-(-6)^n=279937\rightarrow-(-6)^n=279936$$

$$-(-6)^n=(6)^7\ \rightarrow n=7$$

When we add a set number of terms in a geometric sequence, we get a finite geometric series—it starts and stops, plain and simple. But what if we drop the stop sign and keep adding forever?
That's the idea behind an **infinite geometric series**. Let the number of terms head to infinity, and something surprising can happen: under the right conditions, adding up infinitely many numbers can actually give you a nice, finite total.

INFINITE SUM OF A GEOMETRIC SERIES

An infinite geometric series with common ratio r has a finite sum: $S_\infty = \frac{u_1}{1-r}$ if $|r| < 1$

EXAMPLE 7

Determine if the series converges or diverges. $5 - 3 + \frac{9}{5} - \frac{27}{25} + \ldots$

Solution:

it is a geometric series with $r = -\frac{3}{5}$. Since $|r| < 1$ the series converges.

EXAMPLE 8

Determine if the series converges or diverges. $\frac{1}{16} - \frac{1}{4} + 1 - 4 + \ldots$

Solution:

it is a geometric series with $r = -4$. Since $|r| > 1$ the series diverges.

EXAMPLE 9

Evaluate the infinite geometric series described. $u_1 = \frac{9}{5}, r = \frac{1}{2}$

Solution:

Since $|r| < 1$ the series converges.

Using $S_\infty = \frac{u_1}{1-r}$

$$S_\infty = \frac{\frac{9}{5}}{1-\frac{1}{2}} = \frac{\frac{9}{5}}{\frac{1}{2}} = \frac{18}{5}$$

EXAMPLE 10

Evaluate the infinite geometric series described. $0.6+0.12+0.024+0.0048+...$

Solution:

We have $u_1=0.6$ and $r=\dfrac{0.12}{0.6}=\dfrac{1}{5}=0.2$

Since $|r|<1$ the series converges.

Using $S_\infty=\dfrac{u_1}{1-r}=S_\infty=\dfrac{\frac{3}{5}}{1-\frac{1}{5}}=\dfrac{\frac{3}{5}}{\frac{4}{5}}=\dfrac{3}{4}=0.75$

EXAMPLE 11

An arithmetic sequence $a_1,a_2,a_3,...$ and a geometric sequence $u_1,u_2,u_3,...$ both have positive terms. It is given that $a_1=u_1=3$ and $a_3=u_3=27$. Find the value of u_2-a_2, and calculate the ratio of the sum of the first 5 terms of the geometric sequence to the sum of the first 5 terms of the arithmetic sequence.

Solution:

Arithmetic sequence:

$a_1=3$

$a_3=3+2d=27 \rightarrow d=12$

It follows that $a_2=3+12=15$

$S_{a,5}=\dfrac{5}{2}\left[2(3)+4(12)\right]=\dfrac{5}{2}[54]=135$

Geometric sequence:

$u_1=3$

$u_3=3\left(r^2\right)=27 \rightarrow r=3$ (all positive terms)

It follows that $u_2=3(3)=9$

$S_{u,5}=\dfrac{3\left[1-3^5\right]}{1-3}=\dfrac{3[-242]}{-2}=363$

$u_2-a_2=9-15=-6$

Ratio $\dfrac{S_{u,5}}{S_{a,5}}=\dfrac{363}{135}=\dfrac{121}{45}$

EXAMPLE 12

Using the formula for the sum of an infinite geometric series, convert the repeating decimal $0.2\overline{36}=0.236363636...$ into a simplified fraction $\dfrac{p}{q}$.

Solution:

Separate non-repeating and repeating parts:

$$0.2\overline{36} = \frac{2}{10} + \left(\frac{36}{1000} + \frac{36}{100000} + \frac{36}{10000000} + \ldots \right)$$

The terms inside parenthesis form an infinite geometric series with $u_1 = \frac{36}{1000}$ and $r = \frac{1}{100}$

$$S_\infty = \frac{u_1}{1-r} = \frac{\frac{36}{1000}}{1-\frac{1}{100}} = \frac{36}{1000}\left(\frac{100}{99}\right) = \frac{36}{990} = \frac{2}{55}$$

Add with the initial non-repeating term: $0.2\overline{36} = \frac{2}{10} + \left(\frac{2}{55}\right) = \frac{11}{55} + \frac{2}{55} = \frac{13}{55}$

EXAMPLE 13

Find all real values of x for which the infinite geometric series converges, and find the expression for its sum $S(x)$ in terms of x: $\sum_{n=1}^{\infty} 2\left(\frac{2x-1}{3}\right)^n$

Solution:

The first term for $n = 1$ is : $2\left(\frac{2x-1}{3}\right)^1 = \frac{4x-2}{3}$ and $r = \frac{2x-1}{3}$

For convergence, the common ratio must satisfy $|r| < 1$:

$$r = \left|\frac{2x-1}{3}\right| < 1 \;\rightarrow\; -3 < 2x-1 < 3 \;\rightarrow\; -2 < 2x < 4 \;\rightarrow\; -1 < x < 2$$

Using $S_\infty = \frac{u_1}{1-r} = \frac{\frac{4x-2}{3}}{1-\frac{2x-1}{3}} = \frac{\frac{4x-2}{3}}{\frac{3-(2x-1)}{3}} = \frac{4x-2}{4-2x} = \frac{2x-1}{2-x}$

EXAMPLE 14

Evaluate the exact sum of the infinite series: $S = 1 - \tan^2\left(\frac{\pi}{6}\right) + \tan^4\left(\frac{\pi}{6}\right) - \tan^6\left(\frac{\pi}{6}\right) + \ldots$

Solution:

$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ so $\tan^2\left(\frac{\pi}{6}\right) = \frac{1}{3}$

Substituting gives an alternating infinite geometric series: $S = 1 - \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^3 + \ldots$

First term $u_1 = 1$ and $r = -\frac{1}{3}$

$$S_\infty = \frac{u_1}{1-r} = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

EXAMPLE 15

For a geometric sequence with first term u_1 and common ratio $r, 0 < r < 1$, the sum of all its terms is 18, and the sum of the squares of all its terms is 108. Find the first term u_1 and the common ratio r.

Solution:

Original series: $u_1, u_1r, u_1r^2, u_1r^3, \ldots$

Sum: $\dfrac{u_1}{1-r} = 18 \rightarrow u_1 = 18(1-r)$ (equation 1)

Squared series: $(u_1)^2, (u_1r)^2, (u_1r^2)^2, (u_1r^3)^2, \ldots = u_1^2, u_1^2r^2, u_1^2r^4, u_1^2r^6$

Sum: $\dfrac{u_1^2}{1-r^2} = 108 \rightarrow u_1^2 = 108(1-r^2)$ (equation 2). Substitute equation 1 into equation 2:

$$\left[18(1-r)\right]^2 = 108(1-r^2) \quad \rightarrow \quad \frac{324(1-r)^2}{(1-r^2)} = 108$$

$$\frac{324(1-r)^2}{(1-r)(1+r)} = 108 \rightarrow \frac{324(1-r)}{(1+r)} = 108 \rightarrow \frac{(1-r)}{1+r} = \frac{108}{324} = \frac{1}{3}$$

Solving for r:

$$\frac{3(1-r)}{1+r} \rightarrow 3-3r = 1+r \rightarrow 4r = 2 \rightarrow r = \frac{1}{2}$$

Substitute in equation 1:

$$u_1 = 18\left(1-\frac{1}{2}\right) = 9$$

EXAMPLE 16

Find the sum of the first n terms of the series: $S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots nx^{n-1}$ where $x \neq 1$

Solution:

Let $S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots nx^{n-1}$ (equation 1)

Multiply by x: $xS_n = x + 2x^2 + 3x^3 + 4x^4 + \ldots nx^n$ (equation 2)

Subtract equation 2 from equation 1

$$S_n - xS_n = 1 + (2x - x) + (3x^2 - 2x^2) + (4x^3 - 3x^3) + \ldots (nx^{n-1} - (n-1)x^{n-1}) - nx^n$$

$$S_n(1-x) = 1 + x + x^2 + x^3 + \ldots + x^{n-1} - nx^n$$

The sum $1 + x + x^2 + x^3 + \ldots + x^{n-1}$ is a geometric series with first term 1 and $r = x$:

$$S_n(1-x) = \frac{1-x^n}{1-x} - nx^n$$

$$\rightarrow S_n = \frac{\dfrac{1-x^n}{1-x} - nx^n}{(1-x)} = \frac{1-x^n - nx^n(1-x)}{1-x} = \frac{1-x^n - nx^n + nx^{n+1}}{(1-x)^2} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$$

PRACTICE PROBLEMS

Evaluate each geometric series described.

1) $\sum_{n=1}^{10} 4^{n-1}$

349525

2) $\sum_{n=1}^{8} 2^{n-1}$

255

3) $\sum_{m=1}^{9} -2.5 \cdot (-2)^{m-1}$

−427.5

4) $\sum_{n=1}^{8} 3 \cdot (-5)^{n-1}$

−195312

Evaluate the related series of each sequence.

5) −1, −6, −36, −216

−259

6) −3, 6, −12, 24, −48

−33

7) 0.8, 4, 20, 100, 500

624.8

8) −3, −9, −27, −81, −243

−363

Evaluate each geometric series described.

9) $a_1 = -4,\ a_n = -312500,\ r = 5$

−390624

10) $a_1 = -4,\ a_n = \frac{4}{19683},\ r = -\frac{1}{3}$ $-\frac{59048}{19683}$

11) $a_1 = -2,\ a_7 = -8192,\ r = 4$

−10922

12) $a_1 = -3,\ a_7 = -139968,\ r = 6$

−167961

13) $a_1 = -2,\ r = 3,\ n = 8$

−6560

14) $a_1 = 3,\ r = -5,\ n = 8$

−195312

15) $1 - 3 + 9 - 27\ldots,\ n = 6$

−182

16) $-1 - 4 - 16 - 64\ldots,\ n = 8$

−21845

Determine the number of terms *n* in each geometric series.

17) $\sum_{i=1}^{n} 6^{i-1} = 9331$

6

18) $\sum_{m=1}^{n} \left(\frac{1}{3}\right)^{m-1} = \frac{13}{9}$

3

19) $4 - 24 + 144 - 864\ldots,\ S_n = 4444$

5

20) $2 - 10 + 50 - 250\ldots,\ S_n = -5208$

6

21) $\sum_{k=1}^{n} -1080 \cdot \left(\frac{1}{6}\right)^{k-1} = -1290$

3

22) $\sum_{m=1}^{n} (-2)^{m-1} = -21$

6

Determine if the series converges or diverges.

23) $4+2+1+\frac{1}{2}...$

Converges

24) $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}...$

Converges

25) $a_1=-2,\ r=-3$

Diverges

26) $a_1=-\frac{4}{9},\ r=\frac{3}{2}$

Diverges

27) $\sum_{n=1}^{\infty} 5\cdot\left(\frac{1}{4}\right)^{n-1}$

Converges

28) $\sum_{n=1}^{\infty} -3\cdot\left(-\frac{4}{5}\right)^{n-1}$

Converges

Evaluate each infinite geometric series described.

29) $a_1=54,\ r=-\frac{1}{3}$ $\frac{81}{2}$

30) $a_1=-\frac{9}{5},\ r=\frac{1}{3}$ $-\frac{27}{10}$

31) $9.8+5.88+3.528+2.1168...$

24.5

32) $5-1+\frac{1}{5}-\frac{1}{25}...$ $\frac{25}{6}$

33) $\sum_{i=1}^{\infty} -6.4\cdot 0.6^{i-1}$

-16

34) $\sum_{i=1}^{\infty} 5.1\cdot 0.5^{i-1}$

10.2

35) Find the sum of the infinite series: $S=1+\frac{2}{3}+\frac{3}{9}+\frac{4}{27}+\frac{5}{81}+...$ $\frac{9}{4}$

36) For what values of x does the following infinite series converge? $\sum_{n=0}^{\infty}\frac{(2x-3)^n}{5^{n+1}}$

$-1<x<4;\ S=\frac{1}{8-2x}$

37) The first three terms of a sequence are: 4, 12, 36. A second sequence is arithmetic and has the same first term and the same third term. Find the sum of the first 10 terms of both sequences and compare them. $S_{u,10}=118096;\ S_{a,10}=760$

38) A geometric series has $S_5=242$ and $u_1=2$. Find the common ratio r. $r=3$

39) A sequence is defined by: $u_1=48$ and $u_{n+1}=\frac{1}{2}u_n$, Find the infinite sum. 96

40) Find the sum: $2+2(0.4)+2(0.4)^2+2(0.4)^3+...$ Then subtract the first three terms.

$S_\infty=\frac{10}{3}$; Remaining sum $=0.2133$

4.5 APPLICATIONS

You already know how to use the formulas for arithmetic and geometric sequences and series. The next step is knowing which model to use in a real business or economics problem.
The key is to recognize how the quantity changes over time.
Use an arithmetic sequence or series when the amount changes by the same fixed amount each period.
Examples include a fixed annual bonus, straight-line depreciation, or equal principal loan repayments.
Use a geometric sequence or series when the amount changes by the same percentage or factor each period.
Examples include compound interest, inflation, population growth, and investment returns.
Before solving any problem, ask yourself one question:
Is the quantity changing by a fixed amount or by a fixed percentage?
If you can identify the pattern correctly, choosing the right formula becomes much easier.

Arithmetic Sequences: $a_n = a_1 + (n-1)d$ or $a_n = a_{n-1} + d$

Arithmetic Series: $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

Geometric Sequences: $u_n = u_1 r^{n-1}$ or $u_n = u_{n-1} r$

Geometric Series: $S_n = \frac{u_1(1-r^n)}{1-r}$ or $S_\infty = \frac{u_1}{1-r}, \quad |r| < 1$

SIMPLE and COMPOUND INTEREST

Interest is the amount paid for the use of borrowed money, or the amount received for the use of money loaned or invested. There are three factors to consider in calculating interest:
Principal that is the sum loaned or the capital invested.
Time that is the duration of the period.
Rate that is number of units paid upon each hundred units of borrowed sum. The rate of interest can be expressed as an integer or a mixed number with a per cent sign after it, or as a decimal.

SIMPLE INTEREST

Interest calculated on the original principal for the time the principal is used. Simple interest is nothing more than percentage with a time element involved. The original principal remain constant and the quantity of interest for each unit time interval remains constant. The formula for calculating the simple interest:

$$I = Interest = Principal \times Rate \times Time = P \times r \times t$$

EXAMPLE 1

What is the simple interest on \$150 for 2 years at 6% per year?
Solution:
Using the formula $I = Interest = Principal \times Rate \times Time = P \times r \times t$ we have
$I = 150 \times 6\% \times 2 = 18\$$

EXAMPLE 2

Calculate the yearly rate of simple interest needed for an account of 1000EUR to earn a total interest of 240 EUR if invested for 4 years.
Solution:
Using inverse formula and substituting we have:

$$r = rate = \frac{Interest}{Principal \times Time} = \frac{I}{P \times t} = \frac{240}{1000 \times 4} = 0.06 = 6\%$$

EXAMPLE 3

Calculate the time needed to double a principal of 1500EUR invested at a simple interest rate of 4% per annum (p.a)

Solution:

Using inverse formula we have: $t = time = \dfrac{\text{Interest}}{\text{Principal}\times\text{rate}} = \dfrac{I}{P\times r}$. Since the money must double the interest is equals to the principal. Then we have $t = \dfrac{1500}{1500\times 4\%} = 25$.

Therefore it occurs 25 years to double the money.

EXAMPLE 4

If you borrowed $800 at simple interest for 4 years at 5% per annum (p.a.), how much will the creditor receive in all?

Solution:

First we find the interest: $I = 800\times 5\%\times 4 = 160\$$. Therefore the creditor will receive:

Total Amount=Principal+Interest = $800 + 160 = 960\$$

Relation: Arithmetic Sequence and Simple Interest

In example 4, we could have noted that the problem looks like as an arithmetic sequence problem. In fact, if we let:

P_0 = Principal at the beginning

$P_1 = P_0 + I$ = Amount of the Principal after first period

$P_2 = P_1 + I = (P_0 + I) + I = P_0 + 2I$ = Amount of the Principal after second period

…

… in general

$P_t = P_0 + tI$ = Amount of the Principal after t periods

If we look at the similarities with an arithmetic sequence $a_n = a_1 + (n-1)d$ we compare:

$P_0 = a_1$ although the first term in an arithmetic sequence occurs when $n = 1$ while the simple interest the first term occurs at the beginning at time $t = 0$.

The common difference $d = I$ simple interest

.

EXAMPLE 5

Mark must repay a loan of 10,000 EUR from a bank over a period of 5 years. The interest has been included and spread in monthly installments beginning with the first one of 82 EUR, the second one of 86 EUR, the third one of 90 EUR until the last one.

(i) How much will the bank receive in all after 5years?

(ii) How much was the total interest in 5 years?

(iii) Calculate the simple interest rate.

Solution:

In 5 years there are $5\times 12 = 60$ months. Using arithmetic sequence:

$a_1 = 82, d = 4$ and $a_{60} = 82 + (60-1)4 = 318$

then using series: Amount after 5 years $= S_{60} = \dfrac{60}{2}(a_1 + a_{60}) = 30(82 + 318) = 12000\,\text{EUR}.$

Interest = Total Amount-Principal = $2000 EUR$

Using inverse formula and substituting we have:

$$r = rate = \frac{\text{Interest}}{\text{Principal}\times\text{Time}} = \frac{I}{P\times t} = \frac{2000}{10000\times 5} = 0.04 = 4\%$$

COMPOUND INTEREST

In simple interest , the principal remains constant during the term of a loan. In compound interest, the principal is increased by the addition of interest at the end of each interest period during the term of a loan. Whenever the interest is added to the principal at the end of a period, it is said to be compounded. The principal then becomes larger at the beginning of the second period than it was at the beginning of the first period. Basically, the interest due at the end of the second period is larger than that due at the end of the first period. This condition continues for each successive period.

Compound Amount = Principal + Compound Interest

The interest can be compounded annually, semi-annually, quarterly, according to agreement.

In chapter 3, Exponential and Logarithms we have already applied the compound interest formula:

if an amount P_o (called the Principal) is invested at an annual interest rate r compounded n times a year, then in t years the investment will grow to an amount P given by $P_{nt} = P_0 \times \left(1+\frac{r}{n}\right)^{nt}$ *where r is expressed as a decimal.*

EXAMPLE 6

Find the interest for 3 years at 6% on \$1000 with :

(i) simple interest

(ii) compounded annually

Solution:

(i) $I = \text{Interest=Principal}\times\text{Rate}\times\text{Time} = P\times r\times t = 1000\times 6\%\times 3 = 180\$$

(ii) **Method 1**

For 1st year: $I_1 = Interest = 6\%\ of\ P_0 = 6\%\ of\ 1000 = 60\$$

$New\ Principal = P_1 = 1000+60 = 1060\$$

For 2nd year: $I_2 = Interest = 6\%\ of\ P_1 = 6\%\ of\ 1060 = 63.6\$$

$New\ Principal = P_2 = 1060+63.60 = 1123.60\$$

For 3rd year: $I_3 = Interest = 6\%\ of\ 1123.60 = 67.416\$$

$New\ Principal = P_3 = 1123.60+67.416 = 1191.016\$$

Compound interest for 3 years = $P_3 - P_0 = I_1 + I_2 + I_3 = 191.016\$$.

Method 2

Using the formula $P_{nt} = P_0 \times \left(1+\frac{r}{n}\right)^{nt}$ we have $n = 1$ because interest is compounded once a year, $t = 3$, $r = 6\%$, $P_0 = 1000$.

Substituting: $P_3 = 1000\times\left(1+\frac{6\%}{1}\right)^3 = 1000\times(1.06)^3 = 1191.016$

Compound interest for 3 years = $P_3 - P_0 = 1191.016 - 1000 = 191.016\$$.

EXAMPLE 7

A man invests 8000GBP for 12 years at 5% . To the nearest integer, what amount will he get after 12 years if:

(i) interest is simple

(ii) interest is compounded annually

(iii) interest is compounded semi-annually

(iv) interest is compounded quarterly

Solution:

(i) Amount after 12 years = Initial Amount + Interest

Amount after 12 years $= 8000 + (8000 \times 5\% \times 12) = 12{,}800 GBP$

(ii) Using the formula $P_{nt} = P_0 \times \left(1 + \frac{r}{n}\right)^{nt}$ we have $n = 1$ because interest is compounded once a year, $t = 12$, $r = 5\%$, $P_0 = 8000$. Substituting $P_{12} = 8000 \times \left(1 + \frac{5\%}{1}\right)^{12} = 8000 \times (1.05)^{12} = 14{,}367 GBP$

(iii) Using the formula $P_{nt} = P_0 \times \left(1 + \frac{r}{n}\right)^{nt}$ we have $n = 2$ because interest is compounded twice a year, $t = 12$, $r = 5\%$, $P_0 = 8000$. Substituting

$$P_{24} = 8000 \times \left(1 + \frac{5\%}{2}\right)^{24} = 8000 \times (1.025)^{24} = 14{,}470 GBP$$

(iv) Using the formula $P_{nt} = P_0 \times \left(1 + \frac{r}{n}\right)^{nt}$ we have $n = 4$ because interest is compounded four times a year, $t = 12$, $r = 5\%$, $P_0 = 8000$. Substituting

$$P_{48} = 8000 \times \left(1 + \frac{5\%}{4}\right)^{48} = 8000 \times (1.0125)^{48} = 14{,}523 GBP$$

Relation: Geometric Sequence and Compound Interest

As simple interest is connected with arithmetic sequences, compound interest is connected with geometric sequences. In fact, for compound interest we are multiplying by equal amounts after each period. So we can apply geometric sequence and series formula for compound interest problems.

Recall the formula:

the nth term of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1 r^{n-1}$.

Given a geometric sequence with first term a_1, common ratio r and nth term a_n then the sum S_n of the first n terms of the sequence is given by: $S_n = \frac{a_1(1 - r^n)}{1 - r}$, $r \neq 1$

EXAMPLE 8

Jeff has paid university tuitions for his son for 5 years. In the first year, the tuitions were $12,000. They have increased by 6% per annum every year. How much has Jeff paid in total? (to the nearest integer)

Solution:

The tuitions form a geometric sequence with common ratio $r = 1 + 6\% = 1.06$ and first term $a_1 = 12000$ and $n = t = 5$ years.

Using $S_n = \frac{a_1(1 - r^n)}{1 - r}$

$$S_5 = \frac{12000(1 - 1.06^5)}{1 - 1.06} = 67{,}645\$$$

EXAMPLE 9

A company hires a new employee with a starting annual salary of €42,000. The employee receives a raise of €2,200 every year.

1. Write a formula for the salary in the *n-th* year.
2. Find the salary during the 15th year.
3. Determine the total amount paid over the first 15 years.

Solution:

The salary form an arithmetic sequence with $a_1 = 42000$ and $d = 2200$

1. $a_n = 42000 + (n-1)2200$
2. $a_{15} = 42000 + (14)2200 = 72800$
3. $S_{15} = \frac{15}{2}(42000 + 72800) = 861000$

EXAMPLE 10

A machine costs €180,000 and loses 18% of its value every year.

1. Find a formula for its value after *n* years.
2. Find its value after 8 years.
3. During which year will its value first fall below €40,000?

Solution:

We have a geometric sequence with $u_1 = 18000$ and $r = (1 - 0.18) = 0.82$

1. $u_n = 180000(0.82)^7$
2. $u_8 = 180000(0.82)^7 \approx 44910$
3. Solve for *n*:

$180000(0.82)^{n-1} < 40000$

$(0.82)^{n-1} < \frac{40000}{180000} = \frac{2}{9} = 0.\overline{22}$

Take ln:

$n - 1 > \frac{\ln(0.22)}{\ln(0.82)} \approx 7.58 \qquad n = 9$

EXAMPLE 11

A company spends €8,000 on advertising in the first month.
Beginning with month 2:
the advertising budget increases by €600 each month,
sales generated by advertising increase by 6% each month, beginning with €40,000 in month 1.
Find

1. the total advertising cost during the first 12 months,
2. the total sales generated during the first 12 months.

Solution:

1. Advertising: $a_1 = 8000$ and $d = 600$. Apply $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$S_{12} = \frac{12}{2}\left[2(8000) + (11)600\right] = 6[16000 + 6600] = 135600$

2. Sales: $u_1 = 40000$ and $r = 1.6$

$S_{12} = \frac{40000(1 - 1.06^{12})}{1 - 1.06} = \frac{40000(1 - 1.06^{12})}{1 - 1.06} = \frac{40000(1.012)}{0.06} = 674800$

EXAMPLE 12

A borrower agrees to repay a loan in 30 monthly payments.
The first payment is €350, and each payment is €15 larger than the previous one.
Find

1. the final payment,
2. the total amount repaid.

Solution:
Arithmetic: $a_1 = 350,\ d = 15$ and $n = 30$

1. $a_{30} = 350 + (29)15 = 785$
2. $S = \frac{30}{2}[350 + 785] = 15[1135] = 17025$

EXAMPLE 13

A city gains 4,500 new residents every year.At the same time, the city's annual infrastructure budget begins at €20 million and increases by 5% each year. After 15 years determine:

1. the total population increase,
2. the total infrastructure budget spent.

Solution:

1. Population increase: $4500(15) = 67500$
2. Budget: $\frac{20(1.05^{15} - 1)}{1.05 - 1} = \frac{20(2.079)}{0.05} = 1431.6(million)$

EXAMPLE 14

A company purchases a new machine for $120,000. The machine depreciates by 15% of its current value each year. However, the company also spends $5,000 annually on maintenance, which increases by 3% each year due to inflation.
(a) What is the value of the machine after 8 years?
(b) What is the total amount spent on maintenance over the first 8 years?
Solution:
Depreciation is geometric:

$V_n = 120000(0.85)^8 \approx 32698$

we use n = 8 because the machine starts depreciating immediately after purchase.

$V_0 = 120000$ is the starting point, before any depreciation

Maintenance costs form a geometric sequence: $M_n = 5000(1.03)^7 \approx 44461$.

In this case the maintenance start at year 1.

EXAMPLE 15

The inflation rate in the last 6 years has been 4.8% per year. What was the cost of a car (to the nearest EUR)in 2004 if the price in 2009 of the same car is 15,000EUR?
Solution:
Let r = inflation rate
P_0 = price of car in *2004* then P_5 = price of car in *2009* .
we have $P_5 = P_0(1+r)^5$ substitute and solving for P_0

$15000 = P_0(1 + 4.8\%)^5$

$P_0 = \frac{15000}{(1.048)^5} = 11865.46 = 11865EUR$

PRACTICE EXERCISES

1. A company had sales of $50000 during its first year of operation. If the sales increase by $6000 per year, find its sales in the eleventh year. $ 110000

2. John has taken a job with a starting salary of $20000. Find his salary during his fourth year on the job if he receives annual raises of
(i) $1100 $ 23300
(ii) 5% $ 23152.50

3. The number of bacteria in a culture triples every four hours. If 1000 bacteria are present initially, how many bacteria will be present at the end of
(i) 8 hours 9000
(ii) 24 hours 729000

4. A certain machine annually loses 30% of the value it had at the beginning of that year. If its initial value is $10,000 find its value at the end of
(i) the fifth year 1680.70
(ii) (ii) the eight year 823.543

5. A house purchased last year for $80,000 is now worth $96,000. Assuming that the value of the house continues to increase at the same rate each year, find the value 6 years from now. 286654,5

6. Job A has a starting salary of $12,000 with annual increases of $800.
Job B has a starting salary of $11,000 with annual increases of 10%.
Which job will pay more after 3 years? After 5 years?
After 3 years: sum Job A=52800, job B=51051; after 5 years: sum Job A=84000, job B=84872

7. A teacher earns an annual salary of 45 000 USD for the first year of her employment
Her annual salary increases by 1750 USD each year.
(a) Calculate the annual salary for the fifth year of her employment. $45000 + (5 - 1)1750 = 52000$
She remains in this employment for 10 years.
(b) Calculate the total salary she earns in this employment during these 10 years.

$$\frac{10}{2}(2(45000) + (10 - 1)(1750)) = 528750$$

8. A concert choir is arranged, per row, according to an arithmetic sequence. There are 20 singers in the fourth row and 32 singers in the eighth row.
(a) Find the common difference of this arithmetic sequence. $d = 3$
There are 10 rows in the choir and 11 singers in the first row.
(b) Find the total number of singers in the choir. 245

9. A tree begins losing its leaves in October. The number of leaves that the tree loses each day increases by the same number on each successive day.

Date in October	1	2	3	4	.....
Number of leaves lost	24	40	56	72	.....

(a) Calculate the number of leaves that the tree loses on the 21st October. 344
(b) Find the total number of leaves that the tree loses in the 31 days of the month of October. 8184

10. *Give all answers in this question correct to the nearest dollar.*
Clara wants to buy some land. She can choose between two different payment options.
Both options require her to pay for the land in 20 monthly installments.
Option 1: The first installment is $2500. Each installment is $200 more than the one before.
Option 2: The first installment is $2000. Each installment is 800 more than the one before.
(a) If Clara chooses option 1,
(i) write down the values of the second and third installments; 2700$, 2900$

(ii) calculate the value of the final installment; 6300$

(iii) show that the total amount that Clara would pay for the land is $88 000.

(b) If Clara chooses option 2,

(i) find the value of the second installment; 2160$

(ii) show that the value of the fifth installment is $2721.

(c) The price of the land is $80 000. In option 1 her total repayments are $88 000 over the 20 months. Find the annual rate of simple interest that gives this total. 6%

(d) Clara knows that the total amount she would pay for the land is not the same for both options. She wants to spend the least amount of money. Find how much she will save by choosing the cheaper option. 3524$

11. Throughout this question *all* the numerical answers must be given correct to the nearest whole number. Park School started in January 2000 with 100 students. Every full year, there is an increase of 6 % in the number of students.

(a) Find the number of students attending Park School in

(i) January 2001; 106

(ii) January 2003. 119

(b) Show that the number of students attending Park School in January 2007 is 150.

Grove School had 110 students in January 2000. Every full year, the number of students is 10 more than in the previous year.

(c) Find the number of students attending Grove School in January 2003. 140

(d) Find the year in which the number of students attending Grove School will be first 60 % more than in January 2000. n=7, therefore 2007

Each January, one of these two schools, the one that has more students, is given extra money to spend on sports equipment.

(e) (i) Decide which school gets the money in 2007. Justify your answer. 180 Grove School

(ii) Find the first year in which Park School will be given this extra money. n=19, year 2019

12. Two students Ann and Ben play a game. Each time Ann passes GO she receives $15. Each time Ben passes GO he receives 8% of the amount he already has. Both students start with $100.

(a) How much money will Ann have after she has passed GO 10 times? 250

(b) How much money will Ben have after he passes GO 10 times? 215.89

(c) How many times will the students have to pass GO for Ben to have more money than Ann? after 16 years

13. Ann and John go to a swimming pool.
They both swim the first length of the pool in 2 minutes.
The time John takes to swim a length is 6 seconds more than he took to swim the previous length.
The time Ann takes to swim a length is 1.05 times that she took to swim the previous length.

(a) (i) Find the time John takes to swim the third length.
2 minutes + 6 seconds + 6 seconds = 2 minutes 12 seconds

(ii) Show that Ann takes 2.205 minutes to swim the third length. $2(1.05)^2 = 2.205$

(b) Find the time taken for Ann to swim a total of 10 lengths of the pool. 25.2 minutes (or 25 minutes 12 seconds)

14. A National Lottery is offering prizes in a new competition. The winner may choose one of the following.

Option one: $1000 each week for 10 weeks.

Option two: $250 in the first week, $450 in the second week, $650 in the third week, increasing by $200 each week for a total of 10 weeks.

Option three: $10 in the first week, $20 in the second week, $40 in the third week

continuing to double for a total of 10 weeks.

(a) Calculate the amount you receive in the tenth week, if you select

(i) option two; 2050

(ii) option three. 5120

(b) What is the total amount you receive if you select option two? 11500

(c) Which option has the greatest total value? Justify your answer by showing all appropriate calculations. Option One: \$10000; Option Two: \$11500; Option Three: \$10230

15. On Vera's 18th birthday she was given an allowance from her parents. She was given the following choices.

Choice A \$100 every month of the year.

Choice B A fixed amount of \$1100 at the beginning of the year, to be invested at an interest rate of 12% per annum, compounded monthly.

Choice C \$75 the first month and an increase of \$5 every month thereafter.

Choice D \$80 the first month and an increase of 5% every month.

(a) Assuming that Vera does not spend any of her allowance during the year, calculate, for each of the choices, how much money she would have at the end of the year.

Choice A = \$1200; Choice B = \$1239.51; Choice C = \$1230; Choice D = \$1273.37

(b) Which of the choices do you think that Vera should choose? Give a reason for your answer.
Choice D is the highest

(c) On her 19th birthday Vera invests \$1200 in a bank that pays interest at r% per annum compounded annually. Vera would like to buy a scooter costing \$1452 on her 21st birthday. What rate will the bank have to offer her to enable her to buy the scooter? 10%

16. Mr Jones decides to increase the amount of money he spends on food by d GBP every year. In the first year he spends a GBP. In the 8th year he spends twice as much as in the 4th year. In the 20th year he spends 4000 GBP. Find the value of d. d = 200

17. A woman deposits \$100 into her son's savings account on his first birthday. On his second birthday she deposits \$125, \$150 on his third birthday, and so on.

(a) How much money would she deposit into her son's account on his 17th birthday? \$500

(b) How much in total would she have deposited after her son's 17th birthday? \$5100

18. The annual fees paid to a school for the school years 2000, 2001 and 2002 increase as a geometric progression. The table below shows the fee structure.

Year	Fees (USD)
2000	8000.00
2001	8320.00
2002	8652.80

(a) Calculate the common ratio for the increasing sequence of fees. r=1.04

In parts (b) and (c) give your answer correct to 2 decimal places.
The fees continue to increase in the same ratio.

(b) Find the fees paid for 2006. 10122.55

A student attends the school for eight years, starting in 2000.

(c) Find the total fees paid for these eight years. \$73713.81

19. The population of big cats in Africa is increasing at a rate of 5 % per year. At the beginning of 2004 the population was 10 000.

(a) Write down the population of big cats at the beginning of 2005. 10500

(b) Find the population of big cats at the beginning of 2010. 13400

(c) Find the number of years, from the beginning of 2004, it will take the population of big cats to exceed 50 000. n= 33

20. The cost of boring a well 400 meters deep is calculated from the following information: the cost for the first meter is $15.00 and then the cost per meter increases by $3.00 for every subsequent meter. Find

(a) the cost of boring the 300th meter: $912

(b) the total cost of boring the well $245400

21. The Smiths paid $300 for electricity during 1995. If their costs increased 8% each year find:

(i) how much did they pay in 2000 (to the nearest integer) $441

(ii) in what year they pay more than $700 2007

(iii) how much did they pay in total between 1995 and 2009 inclusive (to the nearest integer) $8146

22. Mary is given a test consisting of 15 questions. The first question is worth five points, and each question after the first is worth three points more than the question before it. What is the maximum score that Mary can obtain? 390

23. Given that:

Job A: make $20,000 the first year with annual raises of $1,500

Job B: make $18,000 the first year with annual raise of 10%

(a) Which of the two jobs described below will pay you the higher salary during the fifth year of employment? job B: 26353 vs job A: 26000;

(b) Which will pay you the greater total amount for all five years? job A: 115000 vs 109891

24. A new firm sells 60 solar heating system during its first year in business. Each year after that, it sells 40 more systems than in the preceding year. Find the number of systems that will be sold during the firm's first 10 years in business. 2400

REVIEW CHAPTERS 1-4

1. Simplify: $\dfrac{1}{\csc x-1}+\dfrac{1}{\csc x-1}$ $\left[2\tan x\sec x\right]$

2. If $\dfrac{\pi}{2}<x<\pi$ *and* $\cos x=-\dfrac{3}{5}$ *find* $\sin x+\cos x$ $\left[-\dfrac{8}{15}\right]$

3. Evaluate $4\csc\dfrac{\pi}{3}-2\sec\dfrac{\pi}{6}+3\cot\dfrac{\pi}{4}-\tan\dfrac{5}{6}\pi$ $\left[3+\dfrac{5}{\sqrt{3}}\right]$

4. Solve for $x\in[0,2\pi)$: $\dfrac{\sin x}{1-\cos x}-\tan x=0$ $\left[0,\pi,\dfrac{2}{3}\pi,\dfrac{4}{3}\pi\right]$

5. Solve $\sec^2 x+\tan^2 x=3$ $\left[\dfrac{\pi}{4},\dfrac{3}{4}\pi\dfrac{5}{4}\pi,\dfrac{7}{4}\pi\right]$

6. Find all solution of $\sin^2 x-\cos x-1=0$, $x\in[0,2\pi)$ $[0,\pi]$

7. Simplify $\dfrac{1+\tan x}{\sec x\cos 2x}$ $\left[\dfrac{1}{\cos x-\sin x}\right]$

8. Given an arithmetic sequence where $a_5=\dfrac{7}{2}$ *and* $a_{15}=\dfrac{29}{2}$, *find* a_{23} $\left[\dfrac{43}{2}\right]$

9. Find the sum of the infinite series: $9,-6,+4,-\dfrac{8}{3},\ldots$ $\left[\dfrac{27}{5}\right]$

10. Determine n given $\sum_{k=1}^{n}5k-3=714$ $[17]$

11. Write True or False

    (a) $2\log 5-\log 4=2\log\dfrac{5}{2}$ $[F]$

    (b) $\dfrac{\log a}{\log b}=\log\dfrac{a}{b}$ $[F]$

12. Solve $\log_3\left(\dfrac{27}{x}\right)+\log_3 x^4=6$ $[3]$

13. Solve $\log_2(3x+1)+\log_4(x+3)=\dfrac{5}{2}$ $[5]$

14. Solve: $\dfrac{343}{7^{x-2}}=49$ $[3]$

15. Solve: $16^{x+1}=64^{2x-3}$ $\left[\dfrac{11}{4}\right]$

16. Find the inverse: $f(x)=2+(x-2)^5$ $\left[f^{-1}(x)=\sqrt[5]{x-2}+2\right]$

17. Find the inverse: $f(x)=\log(3\times 5^x)$ $\left[f^{-1}(x)=\log_5\dfrac{10^x}{3}\right]$

18. Write True (T) or False (F)

a. If $a>0$ then $f(x)=ax^2+bx+c$ has a minimum at $x=-\dfrac{b}{2a}$ True

b. If $x+\sqrt{y}$ is a root (zero) of a polynomial $p(x)$ then also $-x-\sqrt{y}$ is a root. False

19. Given $f(x)=\dfrac{x^3+2}{x-1}$ compute $f(2)+f^{-1}(2)$ $[8]$

20. Let $f(x)=10-2x$ and $g(-2)+g^{-1}(2)$ what is the value of
$g(-2)+g^{-1}(2)$ [10]

21. Given that $2^m-2^n=2016$ by showing all work find x and y [$m = 11, n = 5$]

22. Find parabola ax^2+bx+c with vertex $(3,4)$ passing through $(2,6)$ $\left[2x^2-12x+22\right]$

23. Given $x=3$ is a zero of $p(x)=ax^3+bx^2+cx-72$. Find the other roots. $[-4,6]$

24. $(x-1)$ is a factor of $p(x)=ax^4+bx^3+cx^2-43x+60$, leaves reminder -70
when divided by $(x-2)$ and goes through $(-1,80)$. Find a,b,c. $[a=1,b=3,c=-21]$

25. Let $f(x)=3x^5-25x^4+72x^3-72x^2-16x+48$, given that 2 is a root with multiplicity 3,
factorise $f(x)$ completely and state the other roots. $\left[(x-2)^3(3x^2-7x-6),3,-\frac{2}{3}\right]$

26. Given $f(x)=|x+4|$ *and* $g(x)=2x-1$, determine the function $h(x)=f\circ g(x)$ and
Solve $h(x)<7$ $[-5<x<2]$

27. Given $f(x)=5-x^2$ *and* $g(x)=\sqrt{x}+1$ *and* $h(x)=\frac{1}{2}-x$, determine the function
$u(x)=h\circ(g\circ f)$ and find $u(-1)$ $\left[-\frac{5}{2}\right]$

28. Given $f(x)=4-7x$ *and* $g(x)=\frac{x-9}{2}$,

(a) find the value of x such that $f\circ g=g\circ f$

(b) solve $f\left[g(x+1)\right]+f(x)=g\left[f(x)\right]+g(2x)$ $\left[no\ solution,\frac{43}{8}\right]$

29. If $f^{-1}(x)=0$ and $f\circ f(0)=10$ for $f(x)=mx+q$ what is $m+q$? [11]

30. Solve $2^x=16\sqrt{2}$ $\left[\frac{9}{2}\right]$

31. Solve $3^x=\frac{9\sqrt{3}}{\sqrt[4]{3}}$ $\left[\frac{9}{4}\right]$

32. Solve $3^x-3^{x-2}+3^{x+1}=35$ [2]

33. Solve $\frac{2}{3^x-1}=\frac{1}{3^x-5}$ [2]

34. Solve $3\log^2 x-2\log x=0$ $\left[x=1,x=\sqrt[3]{100}\right]$

35. Solve $\log_{4^2} x+3\log\ x=4$ $\left[x=4,x=\frac{1}{256}\right]$

36. Solve $\log_3 x(3\log_3 x-4)+1=0$ $\left[x=\sqrt[3]{3},x=3\right]$

37. Solve $\left(\log_2 x^2\right)^2 + 4\log_2 \sqrt{x} - 2 = 0$ $\left[x = \sqrt{2}, x = \frac{1}{2}\right]$

38. Solve $\log_2\left(\log_7 x\right) = 1$ [49]

39. The solution of $9^x - 3^x - 2 = 0$ is $\frac{\ln a}{\ln b}$. Find *a* and *b*. $[a = 2, b = 3]$

40. Simplify $\frac{1}{1-\cos x} - \frac{1}{1+\cos x}$ $[2\tan x \csc x]$

41. If $x \in \left(\pi, \frac{3}{2}\pi\right)$ and $\cos x = -\frac{5}{13}$, find $\sec x + \tan x$ $\left[-\frac{13}{5}\right]$

42. Evaluate $3\cos\frac{\pi}{3} + 4\tan\frac{\pi}{4} - 2\csc\frac{\pi}{6} + \cot\frac{3\pi}{4}$ $\left[\frac{1}{2}\right]$

43. Solve $\sin 2x = \cos x \quad x \in [0, 2\pi)$ $\left[\frac{\pi}{6}, \frac{\pi}{2}, \frac{5}{6}\pi, \frac{3}{2}\pi\right]$

44. Solve $2\sec x + \tan x = 1 \qquad x \in [0, 2\pi)$ [*no solution*]

45. Simplify $\frac{\sin x}{1+\cos x} + \frac{\sin x}{1-\cos x}$ $[2\csc x]$

46. Given that $f(x) = x^2 - 5x + 2k$ has two equal roots , find the value of *k*. $\left[k = \frac{4}{5}\right]$

47. Given that $f(x) = 2x^2 + 2kx + \left(\frac{3}{2} - k\right)$ has no real roots, find the set of possible values of *k*. $[-3 < k < 1]$

48. Find parabola $y = ax^2 + bx + c$ with vertex $(4, -3)$ through point $(7, 15)$ $\left[2x^2 - 16x + 29\right]$

49. Let $f(x) = ax^3 + (5a-2)x^2 + (4a+2)x - 2a$, given that $(x+3)$ is a factor of $f(x)$ find *a* and factorize $f(x)$ completely. $\left[a = 6, f(x) = 2(x+3)(x+2)(3x-1)\right]$

50. Write True(T) or False(F)

Given the function $y = \frac{x^2+8}{x^2-4}$

(a) The domain is $x \neq \pm 2$ — True
(b) It is positive for $x < -2$ and for $x > 2$ — True
(c) Has a vertical asymptote $x = 1$ — False

(d) Has an horizontal asymptote $x = 2$. False
(e) $x = 2$ is a hole False

51. Write True(T) or False(F)

Given the function $y = \dfrac{2x+1}{x-3}$

(a) The domain of $f(x)$ is all real numbers except $x = 3$. True
(b) The function $f(x)$ has a vertical asymptote at $x = -1/2$ False
(c) The function $f(x)$ has a horizontal asymptote at y = 2. True
(d) The y-intercept of $f(x)$ is -1/3. True
(e) $x = -1$ is a hole False

52. Write a polynomial function of least degree with integral coefficients that has the given zeros:

Zeros: $\dfrac{1}{2}, 2+\sqrt{3}$ $\left[2x^3 - 9x^2 + 6x - 1\right]$

53. Three numbers are in arithmetic sequence. The ratio between the third and the first is 5, while the third exceeds the sum of the first two by 2. Find the three numbers. [2,6,10]

54. The three digits of a number are in arithmetic sequence. By writing the digits in reverse order, you obtain a number that differs from the original by 396. Find the common difference of the sequence. [$d = 2$]

55. Find three positive numbers in arithmetic sequence knowing that their sum is 21 and that the sum of their squares is 165. [4,7,10]

56. Write True (T) or False (F)

a. If $x = a$ is a zero of a function then $x - a$ is a factor of the function True
b. If f and g are polynomials of a degree n, then $f + g$ is a polynomial of degree n. True
c. If $p(x)$ is a polynomial of odd degree with positive leading coefficient then as $x \to -\infty$ then $y \to -\infty$ True

57. Write a polynomial function of least degree with integral coefficients that has the given zeros: $1, \dfrac{3}{2}, -\dfrac{5}{4}$ $\left[8x^3 - 10x^2 - 13x + 15\right]$

58. Given $p(x) = 2x^3 - 3x^2 + 8$, write True or False

a) $p(2) = 12$ [T]
b) $p(-2)$ is the remainder of $p(x)$ divided by $(x-2)$ [F]
c) The remainder of $p(x)$ divided by $(x+1)$ is 3. [F]
d) 8 is the reminder of $p(x)$ divided by x [F]

59. Find the value of k so that $p(x)=x^4-2x^3+kx-1$ is divisible by $(x-2)$ $\left[\frac{1}{2}\right]$

60. $f(x)=x^3+px^2+qx+r$ is divisible by (x^2-1) and leaves a remainder of 12 when divided by $(x-2)$. Find p,q,r. $[p=3,q=-1,r=-3]$

61. There are three positive roots of the polynomial $p(x)=x^3-7x^2+kx-8$. Given that one is twice of another one, find k. $[k=2]$

62. A polynomial $p(x)=2x^4+ax^3+bx^2-13x+c$ has factor $(2x-1)$, remainder - 20 when divided by $(x-1)$ and passes through $(0,12)$. Find a,b,c. $[a=3,b=-24,c=12]$

63. Find what condition the parameter must satisfy for the equality to hold true:

(a) $\cos x=k-2$ $[1\le k\le 3]$

(b) $\sin x=-2a$ $\left[-\frac{1}{2}\le a\le\frac{1}{2}\right]$

(c) $4a\cos x=a+1$ $\left[a\le-\frac{1}{5}\ or\ a\ge\frac{1}{3}\right]$

(d) $(k-1)\sin x=k$ $\left[k\le\frac{1}{2}\right]$

64. Find the value of the indicated function, using the given information.

(a) $\sin x=\frac{7}{25}\ and\ 0<x<\frac{\pi}{2};\cos x=?$ $\left[\frac{24}{25}\right]$

(b) $\cos x=-\frac{4}{25}\ and\ \pi<x<\frac{3\pi}{2};\sin x=?$ $\left[-\frac{3}{5}\right]$

(c) $\sin x=\frac{1}{3}\ and\ \frac{\pi}{2}<x<\pi;\cos x=?$ $\left[-\frac{2\sqrt{2}}{3}\right]$

(d) $\sin x=-\frac{2}{5}\ and\ \frac{3}{2}\pi<x<2\pi;\cos x=?$ $\left[\frac{\sqrt{21}}{5}\right]$

65. Find the value:

(a) $3\sin\frac{\pi}{3}-2\cos\frac{5}{6}\pi+4\sec\frac{\pi}{6}-\csc\frac{7}{6}\pi$ $\left[\frac{5\sqrt{3}}{2}+\frac{8}{\sqrt{3}}+2\right]$

(b) $2\cot\frac{\pi}{4}+3\tan\frac{2}{3}\pi-4\cos\frac{3}{2}\pi+5\csc\frac{11}{6}\pi$ $\left[-8-3\sqrt{3}\right]$

(c) $5\sin\frac{5}{6}\pi+2\cos\frac{4}{3}\pi-3\sec\frac{2}{3}\pi+\csc\frac{3}{4}\pi$ $\left[\frac{15}{2}+\sqrt{2}\right]$

66. Simplify:

(f) $(a\sin x-2\cos x)^2+(a\cos x+2\sin x)^2-4+a^2\sin\frac{\pi}{2}$ $\left[2a^2\right]$

(g) $4-4\sin^2 x+(\cos x-\sin x)^2+2\cos x(\sin x+\cos x)$ $\left[1+6\cos^2 x\right]$

(h) $\sin^2 x+(4\cos x+\sin x)^2+(4\sin x-\cos x)^2+2\cos^2 x$ $\left[18+\cos^2 x\right]$

(i) $\frac{\sin^3 x-\sin x}{\cos x}+2\tan x+\frac{\sin^2 x}{\tan x}$ $\left[2\tan x\right]$

(j) $\frac{1}{2}\cos x+\frac{\tan^2 x}{1+\tan^2 x}-\sin^2 x+\frac{1}{2}\frac{\sin^2 x}{\cos x\tan^2 x}$ $[\cos x]$

(k) $\csc x\ \frac{\cos^2 x+2\sin^2 x-2}{\cot x}$ $[1-\cos x]$

67. Solve for $\leq x<2\pi$

(f) $2\sin^2 x+5\cos x-4=0$ $\left[x=\frac{\pi}{3},\frac{5}{3}\pi\right]$

(g) $\sin x-\sqrt{3}\cos x=0$ $\left[x=\frac{\pi}{3}\right]$

(h) $\sin x+\cos x-1=0$ $\left[x=0,\frac{\pi}{2}\right]$

(i) $\sin^2 x+2\sin x\cos+\cos^2 x=0$ $\left[x=\frac{3}{4}\pi,\frac{7}{4}\pi\right]$

(j) $2\sqrt{3}\sin^2 x-\sin x\cos x+\sqrt{3}\cos^2 x=\sqrt{3}$ $\left[x=0,\pi,\frac{\pi}{6},\frac{7}{6}\pi\right]$

68. Given $f(x)=ax^2+b$ find a and b such that $f(0)=-2$ *and* $f(1)=4$ $[a=6,b=2]$

69. Find the domain:

(a) $f(x)=\frac{1}{x^2-4x}+\frac{3}{\sqrt{x+2}}$ $[x>2,x\neq 4]$

(b) $f(x)=\frac{\sqrt{1-x}}{\sqrt{x+1}}$ $[-1<x\leq 1]$

(c) $f(x)=\sqrt{\frac{2-x}{x+6}}$ $[-6<x\leq 2]$

70. Let $f(x)=x-1$ *and* $g(x)=2x+3$, find $(f\circ g)^{-1}$ $\left[(f\circ g)^{-1}=\frac{x-2}{x}\right]$

71. Given $f(x)=4-7x$ *and* $g(x)=\frac{x-9}{2}$,

solve $f(g(x+1))+f(x)=g(f(x))+g(2x)$ $\left[x=\frac{43}{8}\right]$

72. Find the standard form of parabola with vertex (2, 1)
passing through (-1, 10) $\left[y=x^2-4x+5\right]$

73. Given $f(x)=\frac{x^2+3}{2x-1}$, compute $f(a)+f^{-1}(a)$ *when* $a=4$ [6]

74. If $g^{-1}(5)=0$ and $(g\circ g)(0)=10$ for $g(x)=mx+q$. What is $m+q$? [11]

75. Given that $xf(5^x)+f(5^{-x})=8$, find $f(5)$ [4]

76. Solve:

(a) $2^x+9(2^x)=40$ [2]

(b) $3(4^x)+\frac{7}{4}(4^x)=19\sqrt{2}$ $\left[x=\frac{5}{4}\right]$

(c) $8^{x-1}=\sqrt[3]{2^{x-3}}$ $\left[x=\frac{3}{4}\right]$

(d) $\left(\sqrt{2}\right)^{x}+\left(\sqrt{2}\right)^{x-1}=2\left(\sqrt{2}+1\right)$ [3]

(e) $4^{2x-1}-4^{2x+1}+3\left(2^{4x}\right)=-\frac{3}{2}$ $\left[x=\frac{1}{4}\right]$

(f) $2^{x}-\sqrt{2}=4-2^{\frac{5}{2}-x}$ $\left[\frac{1}{2},2\right]$

(g) $\frac{4}{2^{x}-1}+\frac{3}{2^{x}+1}=5$ $[1]$

(h) $\left(\frac{2}{5}\right)^{x-1}-\left(\frac{5}{2}\right)^{\frac{x-1}{x}}=0$ $[\pm 1]$

77. Write True or False

(c) $\log_2\sqrt{a^{n}}=\frac{n}{2}\log_2 a$ $[T]$

(d) $\log_2 2b=1+\log_2 b$ $[T]$

(e) $\log 3+\log 7-\log 6=\log\frac{7}{2}$ $[T]$

(f) $\log x-\log\left(x-1\right)+\log 5=\log\frac{5x}{x^{2}-1}$ $[F]$

(g) $81^{\log_3 2}=4$ $[F]$

(h) $7^{-1\log_2 2}=\frac{2}{7}$ $[T]$

(i) $\log_{a^2} b^{2}=\log_a b$ $[T]$

(j) $\log_a b=-\log_{\frac{1}{a}} b$ $[T]$

78. Solve:

(a) $\log\left(x+5\right)-\log\left(x+3\right)=\log 4$ $\left[-\frac{7}{3}\right]$

(b) $\log 7+\log\left(x-3\right)-\log x=\log 14-\log\left(x+1\right)$ $\left[2+\sqrt{7}\right]$

(c) $\log_2\left(x^{2}+2x+8\right)=2+\log_2\left(x+2\right)$ $[0,2]$

(d) $\log_3\sqrt{x}\left(\log_3 x+1\right)-2\log_3 x=2$ $\left[\frac{1}{3},81\right]$

(e) $\left(\log_2 x^{2}\right)^{2}+9\log_2 x+2=0$ $\left[\frac{1}{4},\frac{1}{\sqrt[4]{2}}\right]$

(f) $3=\frac{14}{\log_5 x+2}+\frac{4}{\log_5-1}$ $\left[1,5^{5}\right]$

(g) $3\log_{13}\left(5-x\right)+\log_{13}11=\log_{13}\left(x+3\right)+2\log_{13}\left(5-x\right)$ $\left[\frac{13}{3}\right]$

(h) $\frac{3}{\log_2 x\left(1+\log_2 x\right)}=2-\frac{3}{\log_2 x}$ $\left[\frac{1}{2\sqrt{2}},4\right]$

79. The sum of three numbers in arithmetic progression is -27, and the product of the second and third is 54. Find the three numbers. [-12,-9,-6]

80. The sum of the first three terms of an arithmetic sequence is $\frac{15}{2}$, while the sum of their reciprocals is $\frac{37}{30}$. Find the numbers. $\left[2,\frac{5}{2},3 \text{ or } 3,\frac{5}{2},2\right]$

81. The sum of three terms of a geometric sequence is 21, while the difference between the second and the first is 3. Find the three terms [3, 6, 9 or 1,4,16]

82. The sum of three terms of a geometric sequence is 91, while the difference between the third and the second is 42. Find the three terms $\left[7,21,63 \text{ or } \frac{343}{3},-\frac{98}{3},\frac{28}{3}\right]$

83. The product of the sum of three terms of a geometric progression by the difference between the second and the first is $-\frac{26}{3}$, and the ratio between the second term and the common ratio is 3. Determine the three terms. $\left[3,1,\frac{1}{3}\right]$

84. In a geometric sequence, the sum of the second and the eighth term is 390, while the sum of the fourth and the tenth is 1560. Determine the first term and the common ratio $\left[u_1 \pm 3, r = \pm 2\right]$

85. Determine *n* given given4 $\sum_{k=1}^{n} 9-2k = -210$ [30]

86. Find the sum of the infinite series $\frac{18}{5}-\frac{12}{5}+\frac{8}{5}-...$ [6]